von Neuman ordinals:
$\lambda := [0,\lambda)$
Ordinals:
1). 0
2). successor ordinals
3). limit ordinals

The theory is due to Stevo Todorcevic.

We would like to construct complicated combinatorial objects at the level of the second infinite cardinal, ω_1. For this, we first fix a witness to the the fact that all ordinals below ω_1 are countable. That is, we fix a *ladder system* $\vec{C} = \langle C_\alpha \mid \alpha < \omega_1 \rangle$. We require:

Def(Cofinal Subset). $B \subset A$ is a cofinal subset iff $\forall a \in A$, $\exists b \in B$, st $a \leq b$.

- For every $\alpha < \omega_1$, $C_{\alpha+1} = \{\alpha\}$.
- For every limit $\alpha < \omega_1$, C_α is some cofinal subset of α of order-type ω.

order isomorphic to ω.

limit ordinals
λ is a limit ordinal iff $\forall \beta < \lambda$, $\exists \beta < \gamma < \lambda$.

Alternatively, we could require that for each $\alpha < \omega_1$, C_α be a subset of α satisfying $\boxed{\operatorname{otp}(C_\alpha) \leq \omega \text{ and } \sup(C_\alpha) = \sup(\alpha).}$

ω is a limit ordinal.
$\omega+1$ is a successor.

Note that the existence of such a ladder system follows from the Axiom of Choice. Until a rather late stage, the particular choice of the ladder system will have an inessential effect on the outcome objects, and hence will be suppressed.

Given $\alpha < \beta < \omega_1$, we would like to specify a way to *walk* from β down to α.

Let $\beta_0 := \beta$. Clearly, $\beta_0 > \alpha$. Now, for all $n < \omega$ such that β_n is defined and satisfies $\beta_n > \alpha$, we let $\beta_{n+1} := \min(C_{\beta_n} \setminus \alpha)$. Clearly, $\beta_n > \beta_{n+1} \geq \alpha$. Since there cannot be an infinite sequence of decreasing ordinals, there must exist some $n < \omega$, such that $\beta_{n+1} = \alpha$.

Well-ordered

In this course, we shall analyze various functions that record fragments of the information concerning such walks.

Notation. $\boxed{\text{Let } [\omega_1]^2 := \{(\alpha, \beta) \mid \alpha < \beta < \omega_1\}.}$

Functions from w to w_1.

Definition 1.1 (Upper trace function)**.** Define $\operatorname{Tr} : [\omega_1]^2 \to {}^{\omega}\omega_1$ as follows. For all $\alpha < \beta < \omega_1$, let:

The function is strictly decreasing before reaching α.

$$\operatorname{Tr}(\alpha,\beta)(n) := \begin{cases} \beta, & n = 0 \\ \min(C_{\operatorname{Tr}(\alpha,\beta)(n-1)} \setminus \alpha), & n > 0, \ \& \ \operatorname{Tr}(\alpha,\beta)(n-1) > \alpha \\ \alpha, & \text{otherwise.} \end{cases}$$

Lemma 1.2. *For every $\alpha < \beta < \omega_1$ and positive $i < \omega$, there exists a unique $n < \omega$ such that $\operatorname{Tr}(\alpha,\beta)(i) \in C_{\operatorname{Tr}(\alpha,\beta)(n)}$.*[1]

Proof. Write $\tau := \operatorname{Tr}(\alpha,\beta)(i)$. Let $n < \omega$ be the least integer to satisfy $\operatorname{Tr}(\alpha,\beta)(n+1) = \tau$. In particular, $\min(C_{\operatorname{Tr}(\alpha,\beta)(n)} \setminus \alpha) = \tau$ and so $\tau \in C_{\operatorname{Tr}(\alpha,\beta)(n)}$. Next, let $m < \omega$ be arbitrary.

▶ If $m < n$ and $\tau \in C_{\operatorname{Tr}(\alpha,\beta)(m)}$, then by $\alpha \leq \tau$, we would get that

$$\operatorname{Tr}(\alpha,\beta)(m+1) = \min(C_{\operatorname{Tr}(\alpha,\beta)(m)} \setminus \alpha) \leq \tau = \operatorname{Tr}(\alpha,\beta)(n+1).$$

[1] Indeed, $n = i - 1$.

As $m < n$, we must also have $\mathrm{Tr}(\alpha,\beta)(m+1) \geq \mathrm{Tr}(\alpha,\beta)(n+1)$. So $\mathrm{Tr}(\alpha,\beta)(m+1) = \tau$, contradicting the minimality of n.

▶ If $m > n$, then $\mathrm{Tr}(\alpha,\beta)(m) \leq \mathrm{Tr}(\alpha,\beta)(n+1) = \tau$, and hence $C_{\mathrm{Tr}(\alpha,\beta)(m)} \subseteq \tau$, so τ cannot be an element of $C_{\mathrm{Tr}(\alpha,\beta)(m)}$. □ Since $\tau \notin \tau$.

By the preceding lemma, for every $\alpha < \beta < \omega_1$, there is a unique $j < \omega$ such that $\alpha \in C_{\mathrm{Tr}(\alpha,\beta)(j)}$. Let us designate a function that records its successor.

Definition 1.3 (Number of steps function)**.** Define $\rho_2 : [\omega_1]^2 \to \omega$ by stipulating

$$\rho_2(\alpha,\beta) := \min\{n < \omega \mid \mathrm{Tr}(\alpha,\beta)(n) = \alpha\}.$$

For all $\alpha < \beta < \omega_1$, $\mathrm{Tr}(\alpha,\beta)$ is almost constant, so it makes sense to chop it.

Definition 1.4 (Chopped upper trace)**.** Define $\mathrm{tr} : [\omega_1]^2 \to {}^{<\omega}\omega_1$ by stipulating

$$\mathrm{tr}(\alpha,\beta) := \mathrm{Tr}(\alpha,\beta) \restriction \rho_2(\alpha,\beta).$$

$\boxed{\mathrm{dom}(\mathrm{tr}(\alpha,\beta))=\rho_2(\alpha,\beta)}$

$\boxed{\text{Latin: Note well}}$

Nota bene that α is not in the range of $\mathrm{tr}(\alpha,\beta)$.

To streamline some of the arguments, we also define $\mathrm{tr}(\alpha,\alpha) := \emptyset$ and $\rho_2(\alpha,\alpha) := 0$.

Lemma 1.5 (The first concatenation lemma)**.** *For all $\alpha < \beta < \delta < \omega_1$, the following are equivalent:*

(1) $\mathrm{tr}(\alpha,\delta) = \mathrm{tr}(\beta,\delta)^\frown \mathrm{tr}(\alpha,\beta)$;
(2) $\mathrm{Tr}(\alpha,\delta)(\rho_2(\beta,\delta)) = \beta$;
(3) *there exists $m < \omega$ such that* $\mathrm{tr}(\alpha,\delta)(m) = \beta$.

Proof. (1) $\implies$ (2): By $\mathrm{dom}(\mathrm{tr}(\beta,\delta)) = \rho_2(\beta,\delta)$, we have $\rho_2(\beta,\delta) \notin \mathrm{dom}(\mathrm{tr}(\beta,\delta))$, and so hypothesis (1) implies that

$$\mathrm{tr}(\alpha,\delta)(\rho_2(\beta,\delta)) = \mathrm{tr}(\alpha,\beta)(0) = \beta.$$

Therefore, $\mathrm{Tr}(\alpha,\delta)(\rho_2(\beta,\delta)) = \beta$.

(2) $\implies$ (3): $\rho_2(\beta,\delta)$ witnesses the existence of m.

(3) $\implies$ (1): Fix $m < \omega$ such that $\mathrm{Tr}(\alpha,\delta)(m) = \beta$. We first show that $\mathrm{Tr}(\alpha,\delta)(n) = \mathrm{Tr}(\beta,\delta)(n)$ for all $n \leq m$.

Clearly, $\mathrm{Tr}(\alpha,\delta)(0) = \delta = \mathrm{Tr}(\beta,\delta)(0)$. Now, given $n < m$ for which $\mathrm{Tr}(\alpha,\delta)(n) = \mathrm{Tr}(\beta,\delta)(n)$, we call the latter by γ and notice that:

- $\mathrm{Tr}(\alpha,\delta)(n+1) = \min(C_\gamma \setminus \alpha)$, and
- $\mathrm{Tr}(\beta,\delta)(n+1) = \min(C_\gamma \setminus \beta)$.

As $\alpha < \beta$, we have $\min(C_\gamma \setminus \alpha) \leq \min(C_\gamma \setminus \beta)$.

If $\min(C_\gamma \setminus \alpha) < \min(C_\gamma \setminus \beta)$, then the intersection of C_γ with the interval $[\alpha,\beta)$ is nonempty, so that $\mathrm{Tr}(\alpha,\delta)(n+1) = \min(C_\gamma \setminus \alpha) < \beta$.

As $n+1 \leq m$, and $\mathrm{Tr}(\alpha,\delta)(n+1) < \beta$, we then get that $\mathrm{Tr}(\alpha,\delta)(m) < \beta$, contradicting the hypothesis on m. So, $\min(C_\gamma \setminus \alpha) = \min(C_\gamma \setminus \beta)$.

As $\mathrm{Tr}(\alpha,\delta)(m) = \beta$ is not equal to α, $\mathrm{Tr}(\alpha,\delta) \restriction (m+1)$ is injective. So, we have established that $\mathrm{Tr}(\beta,\delta) \restriction (m+1) = \mathrm{Tr}(\alpha,\delta) \restriction (m+1)$ is injective, with $\mathrm{Tr}(\beta,\delta)(m) = \mathrm{Tr}(\alpha,\delta)(m) = \beta$. Consequently, $\rho_2(\beta,\delta) = m$.

So $\mathrm{tr}(\beta,\delta) = \mathrm{Tr}(\beta,\delta) \restriction m = \mathrm{Tr}(\alpha,\delta) \restriction m = \mathrm{tr}(\alpha,\delta) \restriction m$. Finally, as $\mathrm{Tr}(\alpha,\delta)(m) = \beta = \mathrm{Tr}(\alpha,\beta)(0)$, we have $\mathrm{Tr}(\alpha,\delta)(m+i) = \mathrm{Tr}(\alpha,\beta)(i)$ for all $i < \omega$. So, $\mathrm{tr}(\alpha,\delta) = \mathrm{tr}(\beta,\delta)^\frown \mathrm{tr}(\alpha,\beta)$. $\square$

ρ_2 is a two-dimensional function, but it induces one-dimensional functions. Specifically, for each $\delta < \omega_1$, the fiber map $\rho_{2\delta} : \delta \to \omega$ is defined by letting for all $\alpha < \delta$:

$$\boxed{\rho_{2\delta}(\alpha) := \rho_2(\alpha,\delta).}$$

Exercise 1.6. Show that $\rho_{2\delta}$ is order-reversing iff δ is finite.

Exercise 1.7 (The Milner-Rado paradox)**.** Show that for every infinite cardinal κ and every ordinal $\delta < \kappa^+$, there exists a chain $\{D_n \mid n < \omega\}$ such that $\bigcup_{n<\omega} D_n = \delta$ and, for all $n < \omega$, $\mathrm{otp}(D_n) \leq \kappa^n$.

Our next task is showing that ρ_2 gives rise to an Aronszajn tree:

Definition 1.8. A collection $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is said to be an $\aleph_1$-*tree* if all of the following hold:

- $\mathcal{T}$ is downward closed, i.e., for all $f : \delta \to \omega_1$ in $\mathcal{T}$ and $\beta < \delta$, we have $f \restriction \beta$ is in $\mathcal{T}$;
- $\mathcal{T}_\delta := \mathcal{T} \cap {}^\delta\omega_1$ — the δ^{th}-level of $\mathcal{T}$ — is countable, for all $\delta < \omega_1$;
- $\mathcal{T}$ has height ω_1, that is $\{\mathrm{dom}(f) \mid f \in \mathcal{T}\} = \omega_1$.

$\mathcal{T}$ is said to be an $\aleph_1$-*Aronszajn tree*, if, in addition:

- $\mathcal{T}$ has no cofinal branches, i.e., for every $b : \omega_1 \to \omega_1$, there exists some $\delta < \omega_1$ such that $b \restriction \delta \notin \mathcal{T}$.

Now, consider the following subfamily of ${}^{<\omega_1}\omega_1$:

$$\mathcal{T}(\rho_2) := \{\rho_{2\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}.$$

Evidently, $\mathcal{T}(\rho_2)$ is downward closed and has height ω_1. Let us show that its levels are countable.

Lemma 1.9. *For every $\beta < \omega_1$, the set $\{\rho_{2\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$ is countable.*

Proof. Suppose not. Pick $\beta < \omega_1$ and an uncountable $D \subseteq \omega_1 \setminus (\beta+1)$ such that $\rho_{2\delta} \restriction \beta \neq \rho_{2\delta'} \restriction \beta$ for all $\delta < \delta'$ in D. As $\{\rho_2(\beta,\delta) \mid \delta \in D\}$ is a subset of the countable set ω, we may pick an uncountable $D' \subseteq D$ such that $\{\rho_2(\beta,\delta) \mid \delta \in D'\}$ is a singleton, say, $\{n\}$.

For all $\delta \in D'$ and $i < n$, we know that $\mathrm{Tr}(\beta,\delta)(i) > \beta$, and hence $C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta$ is a finite (possibly empty) subset of β.[2] So $\langle C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta \mid i < n\rangle$ is a finite sequence of finite subsets of β. It follows that $\{\langle C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta \mid i < n\rangle \mid \delta \in D'\}$ is countable. Pick an uncountable $D'' \subseteq D'$ such that $\{\langle C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta \mid i < n\rangle \mid \delta \in D''\}$ is a singleton, say, $\{\langle C^i \mid i < n\rangle\}$. Write $C^n := C_\beta$. Then for all $\delta \in D''$, we have:

$$\langle C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta \mid i \leq \rho_2(\beta,\delta)\rangle = \langle C^i \mid i \leq n\rangle.$$

Pick $\delta < \delta'$ in D''. As $\rho_{2\delta} \restriction \beta \neq \rho_{2\delta'} \restriction \beta$, let us fix some $\alpha < \beta$ such that $\rho_{2\delta}(\alpha) \neq \rho_{2\delta'}(\alpha)$. Recall that $\alpha < \beta < \delta < \delta'$ and $C^n = C_\beta$. In particular, $C^n \setminus \alpha$ is nonempty. Now, let $m \leq n$ be the least such that $C^m \setminus \alpha$ is nonempty.

▶ If $m = 0$, then

$$\mathrm{Tr}(\alpha,\delta)(1) = \min(C_\delta \setminus \alpha) = \min(C_\delta \cap \beta \setminus \alpha) = \min(C^0 \setminus \alpha) = \min(C_{\delta'} \cap \beta \setminus \alpha) = \min(C_{\delta'} \setminus \alpha) = \mathrm{Tr}(\alpha,\delta')(1).$$

Consequently, $\mathrm{Tr}(\alpha,\delta)(i+1) = \mathrm{Tr}(\alpha,\delta')(i+1)$ for all $i < \omega$, and hence $\rho_{2\delta}(\alpha) = \rho_{2\delta'}(\alpha)$.

This is a contradiction to the choice of α.

▶ If $m > 0$, then let us first point out that

- $\mathrm{Tr}(\alpha,\delta) \restriction m+1 = \mathrm{Tr}(\beta,\delta) \restriction m+1$, and
- $\mathrm{Tr}(\alpha,\delta') \restriction m+1 = \mathrm{Tr}(\beta,\delta') \restriction m+1$.

We concentrate on δ, as the proof for δ' is the same.

Clearly, $\mathrm{Tr}(\alpha,\delta)(0) = \delta = \mathrm{Tr}(\beta,\delta)(0)$. Now, if $n < m$ and $\mathrm{Tr}(\alpha,\delta)(n) = \mathrm{Tr}(\beta,\delta)(n)$, say it is equal to some γ, then

- $\mathrm{Tr}(\alpha,\delta)(n+1) = \min(C_\gamma \setminus \alpha)$;
- $\mathrm{Tr}(\beta,\delta)(n+1) = \min(C_\gamma \setminus \beta)$.

By $n < m$, we have $C^n \setminus \alpha = \emptyset$. As $C_\gamma \cap \beta = C^n \cap \beta$, we then get that $C_\gamma \cap [\alpha,\beta) = \emptyset$. Altogether,

$$\mathrm{Tr}(\alpha,\delta)(n+1) = \min(C_\gamma \setminus \alpha) = \min(C_\gamma \setminus \beta) = \mathrm{Tr}(\beta,\delta)(n+1),$$

as sought. In particular, we have established that:

- $\mathrm{Tr}(\alpha,\delta)(m) = \mathrm{Tr}(\beta,\delta)(m)$;
- $\mathrm{Tr}(\alpha,\delta')(m) = \mathrm{Tr}(\beta,\delta')(m)$.

[2] Indeed, if $\mathrm{Tr}(\beta,\delta)(i) = \gamma+1$, then $C_{\mathrm{Tr}(\beta,\delta)(i)} = \{\gamma\}$ and by $\beta < \gamma+1$, we have $C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \gamma = \emptyset$. If $\mathrm{Tr}(\beta,\delta)(i)$ is a limit ordinal, say, γ, then $C_{\mathrm{Tr}(\beta,\delta)(i)}$ is a cofinal subset of γ of order-type ω, and just like any initial segment of ω is either finite or the whole set, the initial segment $C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta$ is either finite or equal to $C_{\mathrm{Tr}(\beta,\delta)(i)}$. Since a subset of $\beta < \gamma$ cannot be cofinal in γ, we conclude that $C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta$ is finite.

Finally, since $C^m \setminus \alpha$ is nonempty, we are back in a familiar situation:

$$\mathrm{Tr}(\alpha,\delta)(m+1) = \min(C^m \setminus \alpha) = \mathrm{Tr}(\alpha,\delta')(m+1).$$

Consequently, $\mathrm{Tr}(\alpha,\delta)(i+1) = \mathrm{Tr}(\alpha,\delta')(i+1)$ whenever $m \le i < \omega$, and hence $\rho_{2\delta}(\alpha) = \rho_{2\delta'}(\alpha)$. Again, this is a contradiction to the choice of α. $\square$

Our goal now is to prove that $\mathcal{T}(\rho_2)$ admits no cofinal branch. We shall do so by connecting ρ_2 with the following principle from the paper C. Lambie-Hanson and A. Rinot, **Knaster and friends I: Closed colorings and precalibers**, *Algebra Universalis*, 79(4), Art. 90, 39 pp., 2018.

Definition 1.10. $\mathrm{U}(\kappa,\theta)$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ with the property that for every $A \subseteq \kappa$ of size κ and every $\tau < \theta$, there exists $B \subseteq A$ of size κ satisfying $c``[B]^2 \cap \tau = \emptyset$.

c"[B]² means the image of the coloring c for [B]².

We shall prove that ρ_2 is a witness to $\mathrm{U}(\omega_1,\omega)$ and conclude from it that $\mathcal{T}(\rho_2)$ does not admit a cofinal branch.

Exercise 1.11. Prove that $\mathrm{U}(\omega_1,\omega)$ holds.

Recall that $\rho_2 : [\omega_1]^2 \to \omega$ is defined by stipulating

$$\rho_2(\alpha, \beta) := \min\{n < \omega \mid \mathrm{Tr}(\alpha, \beta)(n) = \alpha\}.$$

In the previous lecture, we proved that

$$\mathcal{T}(\rho_2) := \{\rho_{2\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$$

is an $\aleph_1$-tree. Let us point out that the levels (but to the very bottom) of $\mathcal{T}(\rho_2)$ are infinite.

Lemma 2.1. *Suppose $0 < \beta \leq \delta < \omega_1$ and $0 < n < \omega$. Then $\rho_{2\delta} \restriction \beta \neq \rho_{2(\delta+n)} \restriction \beta$.*

Proof. This follows immediately from the next lemma. $\square$

Lemma 2.2. *For all $\alpha < \delta < \omega_1$ and $n < \omega$, we have $\rho_{2(\delta+n)}(\alpha) = \rho_{2\delta}(\alpha) + n$.*

Proof. By induction on $n < \omega$. The case $n = 0$ is trivial. Now, given $n < \omega$ for which $\rho_{2\delta+n}(\alpha) = \rho_{2\delta}(\alpha) + n$, by $C_{\delta+n+1} = \{\delta + n\}$, we have that $\mathrm{Tr}(\alpha, \delta + n + 1)(1) = \min(C_{\delta+n+1} \setminus \alpha) = \delta + n$, and so by the first concatenation lemma (of the previous lecture):

- $\mathrm{tr}(\alpha, \delta + n + 1) = \mathrm{tr}(\delta + n, \delta + n + 1)^\frown \mathrm{tr}(\alpha, \delta + n)$, and
- $\rho_2(\alpha, \delta + n + 1) = 1 + \rho_2(\alpha, \delta + n)$.

By the hypothesis, then, $\rho_{2\delta+n+1}(\alpha) = \rho_{2\delta+n}(\alpha)+1 = \rho_{2\delta}(\alpha)+n+1$. $\square$

We now introduce another characteristic function of the walk.

Definition 2.3. Define $\lambda : [\omega_1]^2 \to \omega_1$ by stipulating:

$$\boxed{\lambda(\beta, \delta) := \max\{\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) \mid i < \rho_2(\beta, \delta)\}.}$$

λ(β, δ) is the maximum of superums of all the elements below β in every ladder.

Lemma 2.4. *Suppose $0 < \beta < \delta < \omega_1$. Then $\lambda(\beta, \delta) < \beta$.*

Proof. For all $i < \rho_2(\beta, \delta)$, $C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta$ is a finite subset of β. So $\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) < \beta$. Consequently, $\lambda(\beta, \delta)$ is the maximum over a finite subset of β, and hence $< \beta$. $\square$

Lemma 2.5 (The second concatenation lemma)**.** *Suppose $\lambda(\beta, \delta) < \alpha < \beta < \delta < \omega_1$. Then $\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\beta, \delta)^\frown \mathrm{tr}(\alpha, \beta)$.*

Proof. By the first concentration lemma, it suffices to prove that

$$\mathrm{tr}(\alpha, \delta)(\rho_2(\beta, \delta)) = \beta.$$

For this, we shall prove that $\mathrm{Tr}(\alpha, \delta)(n) = \mathrm{Tr}(\beta, \delta)(n)$ for all $n \leq \rho_2(\beta, \delta)$.

▶ Clearly, $\mathrm{Tr}(\alpha, \delta)(0) = \delta = \mathrm{Tr}(\beta, \delta)(0)$.

▶ Next, given $n < \rho_2(\beta, \delta)$ for which $\mathrm{Tr}(\alpha, \delta)(n) = \mathrm{Tr}(\beta, \delta)(n)$, say, it is equal to γ, we have:

- $\mathrm{Tr}(\alpha,\delta)(n+1)=\min(C_\gamma\setminus\alpha)$, and
- $\mathrm{Tr}(\beta,\delta)(n+1)=\min(C_\gamma\setminus\beta)$.

Trivially, $C_\gamma\cap(\sup(C_\gamma\cap\beta),\beta)=\emptyset$. As $\sup(C_\gamma\cap\beta)=\sup(C_{\mathrm{Tr}(\beta,\delta)(n)}\cap\beta)\le\lambda(\beta,\delta)<\alpha$, we get in particular that $C_\gamma\cap[\alpha,\beta)=\emptyset$. So $\min(C_\gamma\setminus\alpha)=\min(C_\gamma\setminus\beta)$, and hence $\mathrm{Tr}(\alpha,\delta)(n+1)=\mathrm{Tr}(\beta,\delta)(n+1)$. $\square$

Complementary to Lemma 2.2, we have the following.

Corollary 2.6 (ℓ_∞-coherence)**.** *For all $\beta<\delta<\omega_1$, there exists a large enough $k<\omega$ such that $\sup_{\alpha<\beta}|\rho_{2\beta}(\alpha)-\rho_{2\delta}(\alpha)|\le k$.*

Proof. Suppose that $\beta<\delta<\omega_1$ forms a counterexample. Then, for every $k<\omega$, we can pick $\alpha_k<\beta$ such that $|\rho_{2\beta}(\alpha_k)-\rho_{2\delta}(\alpha_k)|>k$. Define $c:[\omega]^2\to 2$ by letting, for all $n<m<\omega$, $c(n,m):=0$ iff $\alpha_n\le\alpha_m$. By Ramsey's theorem, there exists an infinite $H\subseteq\omega$ which is homogeneous for c. Since there cannot be a strictly decreasing sequence of ordinals, H must be 0-homogeneous. So $\langle\alpha_k\mid k\in H\rangle$ is infinite and non-decreasing. Since $\sup_{k\in H}|\rho_{2\beta}(\alpha_k)-\rho_{2\delta}(\alpha_k)|=\omega$, we may pass to an infinite subset of H, and moreover assume that $\langle\alpha_k\mid k\in H\rangle$ is increasing.

Let $\alpha^*:=\sup\{\alpha_k\mid k\in H\}$. Consider the positive integer

$$n:=|\rho_2(\alpha^*,\delta)-\rho_2(\alpha^*,\beta)|.$$

Let

$$\lambda:=\begin{cases}\lambda(\alpha^*,\delta), & \text{if } \alpha^*=\beta,\\ \max\{\lambda(\alpha^*,\delta),\lambda(\alpha^*,\beta)\}, & \text{if } \alpha^*<\beta.\end{cases}$$

As $\lambda<\alpha^*$, let us pick a large enough $k\in H$ with $k>n$ such that $\alpha_k>\lambda$.

By Lemma 2.5, and as $\lambda(\alpha^*,\delta)\le\lambda<\alpha_k<\alpha^*<\delta$, we have:

$$\mathrm{tr}(\alpha_k,\delta)=\mathrm{tr}(\alpha^*,\delta)^\frown\mathrm{tr}(\alpha_k,\alpha^*).$$

Next, we analyze $\mathrm{tr}(\alpha_k,\beta)$. There are two cases to consider:

▶ If $\alpha^*<\beta$, then by Lemma 2.5, and as $\lambda(\alpha^*,\beta)\le\lambda<\alpha_k<\alpha^*<\beta$, we have:

$$\mathrm{tr}(\alpha_k,\beta)=\mathrm{tr}(\alpha^*,\beta)^\frown\mathrm{tr}(\alpha_k,\alpha^*).$$

▶ If $\alpha^*=\beta$, then $\mathrm{tr}(\alpha^*,\beta)=\emptyset$, and trivially

$$\mathrm{tr}(\alpha_k,\beta)=\mathrm{tr}(\alpha^*,\beta)^\frown\mathrm{tr}(\alpha_k,\alpha^*).$$

Altogether, $|\rho_{2\delta}(\alpha_k)-\rho_{2\beta}(\alpha_k)|=|\rho_2(\alpha^*,\delta)-\rho_2(\alpha^*,\beta)|=n<k$, contradicting the very choice of α_k. $\square$

We now come back to the question of whether $\mathcal{T}(\rho_2)$ is an $\aleph_1$-Aronszajn tree.

Lemma 2.7. *If $\mathcal{T}(\rho_2)$ admits a cofinal branch, then there exists an uncountable $X \subseteq \omega_1$ and $k < \omega$ such that $\rho_2(\alpha, \beta) = k$ for any pair $\alpha < \beta$ of ordinals from X.*

Proof. Suppose that $b : \omega_1 \to \omega$ is a cofinal branch through $\mathcal{T}(\rho_2)$. For each $\beta < \omega_1$, as $b \restriction \beta \in T_\beta$, there exists some countable ordinal $\delta_\beta \geq \beta$ such that $b \restriction \beta = \rho_{2\delta_\beta} \restriction \beta$. In particular, we may recursively define a continuous (increasing) function $f : \omega_1 \to \omega_1$ satisfying the following:[1]

- $f(0) = 0$;
- $b \restriction (f(\alpha) + 1) = \rho_{2f(\alpha+1)} \restriction (f(\alpha) + 1)$.

By the pigeonhole principle, let X be an uncountable subset of $\{f(\eta+1) \mid \eta < \omega_1\}$ on which b is constant, with value, say, k. Let $\alpha < \beta$ be an arbitrary pair of ordinals of X. We shall show that $\rho_2(\alpha, \beta) = k$.

Pick a pair $\gamma < \delta < \omega_1$ such that $\alpha = f(\gamma+1)$ and $\beta = f(\delta+1)$. As $\gamma + 1 \leq \delta$, we have

$$b \restriction (\alpha + 1) = b \restriction (f(\gamma + 1) + 1) \subseteq b \restriction (f(\delta) + 1) \subseteq \rho_{2f(\delta+1)} = \rho_{2\beta}.$$

In particular, $\rho_2(\alpha, \beta) = \rho_{2\beta}(\alpha) = b(\alpha) = k$. □

So, a natural strategy to prove that $\mathcal{T}(\rho_2)$ does not admit a cofinal branch would be to prove that $\rho_{2\beta}$ is finite-to-one for all $\beta < \omega_1$. Unfortunately, this is not the case:

Exercise 2.8. Show that for every nonzero limit $\beta < \omega_1$, $\rho_{2\beta}^{-1}\{1\}$ is infinite.

As promised in the previous lecture, we will prove that ρ_2 witnesses $\mathrm{U}(\omega_1, \omega)$. By Lemma 2.7, we would then conclude that $\mathcal{T}(\rho_2)$ admits no cofinal branch, so that, altogether, is an $\aleph_1$-Aronszajn tree.

A club is a "club" of closure points of some func.

The proof of the next lemma assumes familiarity with *club* and *stationary sets*, which we now introduce in brief. Recall that $\beta < \omega_1$ is said to be a *closure point* of a function $f : \omega_1 \to \omega_1$, if $f(\alpha) < \beta$ for all $\alpha < \beta$. That is, $f[\beta] \subseteq \beta$. Now, a set $C \subseteq \omega_1$ is said to be a *club* if it is the set of closure points of some function $f : \omega_1 \to \omega_1$. That is, $C = \{\beta < \omega_1 \mid f[\beta] \subseteq \beta\}$. For instance, the set of all limit ordinals below ω_1 is a club, as witnessed by the map $\alpha \mapsto \alpha + 1$. A subset $S \subseteq \omega_1$ is said to be *stationary* if it meets any club, that is, if for every $f : \omega_1 \to \omega_1$, there exists some $\beta \in S$ with $f[\beta] \subseteq \beta$. It is easy to see that the the intersection of two clubs is a club, and that the intersection of countably many clubs contains a club. In particular, any club is stationary, and the intersection of a club and a stationary

$(S\cap C)\cap D)=$
$S\cap(C\cap D)$

[1]Since we require the function to be continuous, we only have to specify what to do at the base and successor stages.

set is again stationary. We point out that there is a notion of *c.u.b.* which is closely related to that of a *club*. A subset $C \subseteq \omega_1$ is said to be *closed* if for any increasing $\langle \alpha_n \mid n < \omega \rangle$ of elements from C, the limit $\sup_{n<\omega} \alpha_n$ is in C. A set $C \subseteq \omega_1$ is *unbounded* if for all $\alpha < \omega_1$, there exists $\beta \in C$ with $\alpha < \beta$. Any club set is c.u.b., and any c.u.b. contains a club. In particular, the collection of all clubs and the collection of all c.u.b.'s generate the same filter.

Exercise 2.9. Show that clubs and stationary sets are necessarily uncountable.

An instrumental generalization of the Pigeonhole Principle is *Fodor's lemma* that asserts that for every stationary $S \subseteq \omega_1$ and every $g : S \to \omega_1$ that satisfies $g(\alpha) < \alpha$ for all nonzero $\alpha \in S$, there exists some stationary $T \subseteq S$ for which $g \restriction T$ is constant.

Notation. For sets of ordinals a, b, write $a < b$ whenever $\alpha < \beta$ for all $\alpha \in a$ and $\beta \in b$.

Lemma 2.10. *ρ_2 witnesses that* $\mathrm{U}(\omega_1, \omega, \mathit{finite})$ *holds. That is, for every uncountable family X of pairwise disjoint finite subsets of ω_1, and every $k < \omega$, there exists an uncountable $Y \subseteq X$ such that* $\min(\rho_2[a \times b]) \geq k$ *for all $a < b$ both from Y.*

Proof. The statement is trivially valid for $k = 0$. Next, suppose that the statement is valid for k, and let us prove it for $k + 1$. Since X is uncountable and consists of pairwise disjoint sets, for each $\beta < \omega_1$, we may pick some $x_\beta \in X$ with $\min(x_\beta) > \beta$. Define a function $f : \omega_1 \to \omega_1$ by stipulating $f(\beta) := \max(x_\beta)$, and consider the club $C := \{\beta < \omega_1 \mid f[\beta] \subseteq \beta\}$, so that for any pair of ordinals $\beta < \beta'$ of C, we have $\{\beta\} < x_\beta < \{\beta'\} < x_{\beta'}$.

Next, for all nonzero limit ordinal $\beta \in C$, put $\lambda_\beta := \max\{\lambda(\beta, \alpha) \mid \alpha \in x_\beta\}$. By Lemma 2.4, $\lambda_\beta < \beta$. By Fodor's lemma, let us pick a stationary subset $S \subseteq C$ such that $\{\lambda_\beta \mid \beta \in S\}$ is a singleton, say $\{\lambda\}$, and put $X' := \{\{\beta\} \cup x_\beta \mid \beta \in S\}$. By $S \subseteq C$, we know that X' consists of pairwise disjoint finite sets, so we may appeal to the induction hypothesis and find an uncountable $Y' \subseteq X'$ such that $\min(\rho_2[a \times b]) \geq \kappa$ whenever $a < b$ are both from Y'. Let $Y := \{y \setminus \{\min(y)\} \mid y \in Y'\}$, so that Y is an uncountable subset of X.

Now, let $a < b$ be arbitrary pair of elements of Y. Pick arbitrary $\alpha \in a$ and $\alpha' \in b$, and let us show $\rho_2(\alpha, \alpha') \geq k + 1$. Fix $\beta < \beta'$ such that $a = x_\beta$ and $b = x_{\beta'}$. Since

- $\beta, \beta' \in S$,
- $\alpha \in x_\beta$,

- $\beta' \in C$, and
- $\alpha' \in x_{\beta'}$,

we have
$$\lambda(\beta', \alpha') \leq \lambda_{\beta'} = \lambda = \lambda_\beta < \beta < \alpha < \beta' < \alpha'.$$
So, by Lemma 2.5, we get that $\mathrm{tr}(\alpha, \alpha') = \mathrm{tr}(\beta', \alpha')^\frown \mathrm{tr}(\alpha, \beta')$, and hence $\rho_2(\alpha, \alpha') \geq 1 + \rho_2(\alpha, \beta')$. Finally, as $\{\beta\} \cup a$ and $\{\beta'\} \cup b$ are in Y', and $\alpha \in a$, we have $\rho_2(\alpha, \beta') \geq k$. Altogether, $\mathrm{tr}(\alpha, \alpha') \geq k+1$. $\square$

Corollary 2.11. *$\mathcal{T}(\rho_2)$ is an Aronszajn tree.* $\square$

Exercise 2.12 (Lázár's *Free Set Lemma*, 1936)**.** Prove that for every function $g : \omega_1 \to [\omega_1]^{<\omega}$, there exists an uncountable $F \subseteq \omega_1$ such that, for all $\alpha \neq \beta$ from F, $\alpha \notin g(\beta)$.

Recalling Lemma 2.7, to show that $\mathcal{T}(\rho_2)$ does not admit a branch, it suffices to show that for every uncountable $X \subseteq \omega_1$, ρ_2 is not constant over $[X]^2$. However, in Lemma 2.10, we proved that for every uncountable $X \subseteq \omega_1$, ρ_2"$[X]^2$ is in fact infinite. This fact leads to the stronger conclusion that $\mathcal{T}(\rho_2)$ does not even admit a modulo-finite branch. Let us make it precise:

Corollary 2.13. *For every $b : \omega_1 \to \omega$, there exists $\beta < \omega_1$ such that*
$$\sup_{\alpha<\beta} |\rho_{2\beta}(\alpha) - b(\alpha)| = \infty.$$

Proof. Suppose not. Pick $b : \omega_1 \to \omega$ such that, for all $\beta < \omega_1$, the following is an integer:
$$n_\beta := \sup_{\alpha<\beta} |\rho_{2\beta}(\alpha) - b(\alpha)|.$$

Fix an uncountable $B \subseteq \omega_1$ on which $\{n_\beta \mid \beta \in B\}$ is a singleton, say, $\{n\}$. Then, fix an uncountable $A \subseteq B$ on which $\{b(\alpha) \mid \alpha \in A\}$ is a singleton, say, $\{m\}$. By Lemma 2.10, there exists a pair $\alpha < \beta$ of elements of A such that $\rho_2(\alpha, \beta) \geq n + m + 1$.

► As $\alpha \in A$, we have $b(\alpha) = m$.
► As $\beta \in B$ and $\alpha < \beta$, we have $|\rho_{2\beta}(\alpha) - b(\alpha)| \leq n$.

Altogether, $\rho_2(\alpha, \beta) \leq n + m$, which is a contradiction. $\square$

R-embeddable tree: order function to R.

Definition 3.1. An $\aleph_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is said to be *$\mathbb{R}$-embeddable* iff there exists an order-preserving mapping $f : \mathcal{T} \to \mathbb{R}$ from the tree to the real line.[1]

Aleph$_1$ tree is A-tree if it is R-embeddable.

Lemma 3.2. *If an $\aleph_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is $\mathbb{R}$-embeddable, then $\mathcal{T}$ is Aronszajn.*

Proof. Suppose that $f : \mathcal{T} \to \mathbb{R}$ is order-preserving. Suppose towards a contradiction that $b : \omega_1 \to \omega_1$ is a cofinal branch through $\mathcal{T}$. Since f is order-preserving, for all $\alpha < \omega_1$, we have $f(b \restriction \alpha) < f(b \restriction \alpha + 1)$, so let us pick a rational number q_α such that

$$f(b \restriction \alpha) < q_\alpha < f(b \restriction \alpha + 1).$$

Now, for every pair of countable ordinals $\alpha < \beta$, we have

$$f(b \restriction \alpha) < q_\alpha < f(b \restriction \alpha + 1) \leq f(b \restriction \beta) < q_\beta.$$

So, the mapping $\alpha \mapsto q_\alpha$ is an injection from ω_1 to $\mathbb{Q}$, contradicting the fact that $|\omega_1| > |\mathbb{Q}|$. $\square$

A subset A of ${}^{<\omega_1}\omega_1$ is said to be an *antichain* iff for all two distinct σ, σ' from A, $\sigma \not\subseteq \sigma'$ and $\sigma' \not\subseteq \sigma$.

Definition 3.3. An $\aleph_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is said to be *special* if $\mathcal{T}$ may be covered by countably many antichains.

Exercise 3.4. Show that any special $\aleph_1$-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is $\mathbb{R}$-embeddable, and hence Aronszajn.

It is consistent (in fact, follows from Martin's Axiom + the negation of the Continuum Hypothesis) that all $\aleph_1$-Aronszajn trees are special. It is also consistent that there exists an $\aleph_1$-Aronszajn tree which is not $\mathbb{R}$-embeddable. For instance, there consistently exists a *distributive Aronszajn tree*; such a tree is Aronszajn but it admits a cofinal branch in some outer universe that has the very same set of reals, thereby, colliding with Lemma 3.2. For a recent treatment of the topic, see A. M. Brodsky and A. Rinot, **Distributive Aronszajn trees**, *Fund. Math.*, 245(3): 217-291, 2019.

Exercise 3.5. Show that $\mathcal{T}$ is special iff there exists a countable set Υ, and a function $c : \mathcal{T} \to \Upsilon$ such that the preimage of any singleton is an antichain.

We would like to show that $\mathcal{T}(\rho_2)$ is special. For this, we shall need the following.

[1] Note: f is not required to be one-to-one. It is only required to be one-to-one over chains. Therefore, a better name would have probably been $\mathbb{R}$-*reducible*.

Lemma 3.6. *For all $n < \omega$, $\{\rho_{2(\delta+n)} \mid \delta < \omega_1$ limit nonzero$\}$ is an antichain.*

Idea of the Proof: if one is contained by the other, then all the images of this funs is equal to the other's images. So we just need to find an α such that $\rho_{2\delta}(\alpha) \neq \rho_{2\beta}(\alpha)$

Proof. Let $\beta < \delta$ be pair of arbitrary nonzero limit counatble ordinals. As we have seen last week, $\lambda := \lambda(\beta, \delta)$ is $< \beta$. As β is a limit ordinal, we may fix α with $\lambda < \alpha < \beta$. So by the second concatenation lemma, $\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\beta, \delta)^\frown \mathrm{tr}(\alpha, \beta)$, and hence $\rho_{2\delta}(\alpha) > \rho_{2\beta}(\alpha)$. Consequently, $\rho_{2(\delta+n)}(\alpha) = \rho_{2\delta}(\alpha) + n > \rho_{2\beta}(\alpha) + n = \rho_{2(\beta+n)}(\alpha)$ for all $n < \omega$. □

It follows that $\{\rho_{2\delta} \mid \delta < \omega_1\}$ is the countable union of antichains. However, this does not immediately imply that $\mathcal{T}(\rho_2)$ is special.

Lemma 3.7. *For every pair of countable ordinals $\beta < \delta$, the following set is closed below β:*

$$A := \{\alpha < \beta \mid \rho_{2\beta}(\alpha) = \rho_{2\delta}(\alpha)\}.$$

Proof. Suppose α^* is an ordinal satisfying $0 < \alpha^* = \sup(A \cap \alpha^*) < \beta$. We must show that $\alpha^* \in A$.

Let $\lambda := \max\{\lambda(\alpha^*, \delta), \lambda(\alpha^*, \beta)\}$, so that $\lambda < \alpha^*$. Fix a large enough $\alpha \in A$ above λ. Then, by the second concatenation lemma:

- $\mathrm{tr}(\alpha, \beta) = \mathrm{tr}(\alpha^*, \beta)^\frown \mathrm{tr}(\alpha, \alpha^*)$, and
- $\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\alpha^*, \delta)^\frown \mathrm{tr}(\alpha, \alpha^*)$.

So $\rho_2(\alpha^*, \beta) = \rho_2(\alpha, \beta) - \rho_2(\alpha, \alpha^*) = \rho_2(\alpha, \delta) - \rho_2(\alpha, \alpha^*) = \rho_2(\alpha^*, \delta)$. □

In his 2013 dissertation, Peng introduced a mapping $\pi : {}^{<\omega_1}\omega \to {}^{<\omega_1}\mathbb{Z}$ that exploits Lemma 3.7. Given $f : \beta \to \omega$, let $\pi(f) : \beta \to \mathbb{Z}$ be the unique function that satisfies for all $\alpha < \beta$:

$$\pi(f)(\alpha) = f(\alpha) - f(\sup(\alpha)).$$

That is, we let $\pi(f)(\alpha) := 0$ for all limit $\alpha < \beta$, and let $\pi(f)(\alpha) := f(\alpha) - f(\alpha - 1)$ for all successor $\alpha < \beta$. Write

$$\mathcal{T}_\pi(\rho_2) := \{\pi(f) \mid f \in \mathcal{T}(\rho_2)\}.$$

From now on, we add the assumption that for every limit nonzero ordinal δ, 0 is in its ladder C_δ, so that $\rho_{2\delta}(0) = 1$.

Lemma 3.8 (Peng)**.** *$(\mathcal{T}(\rho_2), \subseteq)$ and $(\mathcal{T}_\pi(\rho_2), \subseteq)$ are order-isomorphic. In particular, $\mathcal{T}(\rho_2)$ is special iff $\mathcal{T}_\pi(\rho_2)$ is special.*

Proof. For all $\alpha < \beta < \omega_1$, and $f : \beta \to \omega$, the value of $\pi(f)(\alpha)$ depends only on $f \restriction (\alpha + 1)$. In particular, $f \subseteq g$ entails $\pi(f) \subseteq \pi(g)$ for any $f, g \in \mathcal{T}(\rho_2)$.

Next, suppose that we are given $f, g \in \mathcal{T}(\rho_2)$ with $f \nsubseteq g$. We shall show that $\pi(f) \nsubseteq \pi(g)$. As $f, g \in \mathcal{T}(\rho_2)$, let us pick $\beta \leq \delta < \omega_1$ and

$\beta' \leq \delta' < \omega_1$ such that $f = \rho_{2\delta} \restriction \beta$ and $g = \rho_{2\delta'} \restriction \beta'$. Now, there are two cases to consider:

▶ If $\beta > \beta'$, then $\mathrm{dom}(\pi(f)) > \mathrm{dom}(\pi(g))$, so that $\pi(f) \not\subseteq \pi(g)$.

▶ Suppose that $\beta \leq \beta'$, and let $\alpha < \beta$ be the least such that $f(\alpha) \neq g(\alpha)$. By Lemma 3.7 and as $\rho_{2\delta}(0) = 1 = \rho_{2\delta'}(0)$, α is a successor ordinal, so that $\pi(f)(\alpha) = f(\alpha) - f(\alpha-1)$ and $\pi(g)(\alpha) = g(\alpha) - g(\alpha-1)$. Now, by $f(\alpha) \neq g(\alpha)$ and $f(\alpha - 1) = g(\alpha - 1)$, we conclude that $\pi(f)(\alpha) \neq \pi(g)(\alpha)$, as sought. □

coherence means: almost all the common domain of every two functions has the same value.

Definition 3.9. A family $\mathcal{T}$ of functions is called *coherent* if for all $f, g \in \mathcal{T}$, the set $\{i \in \mathrm{dom}(f) \cap \mathrm{dom}(g) \mid f(i) \neq g(i)\}$ is finite.

Exercise 3.10. Prove that there exists $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ which is a coherent Aronszajn tree, consisting of injections.

Lemma 3.11 (Peng)**.** *$\mathcal{T}_\pi(\rho_2)$ is coherent.*

Proof. Suppose not. Pick ordinals $\beta \leq \delta < \omega_1$ and $\beta' \leq \delta' < \omega_1$ such that the following set is infinite:

$$\{\alpha < \min\{\beta, \beta'\} \mid \pi(\rho_{2\delta} \restriction \beta)(\alpha) \neq \pi(\rho_{2\delta'} \restriction \beta')(\alpha)\}.$$

In particular,

$$A := \{\alpha < \min\{\delta, \delta'\} \mid \pi(\rho_{2\delta})(\alpha) \neq \pi(\rho_{2\delta'})(\alpha)\}$$

is infinite. Recalling the definition of π, A consists of successor ordinals. Let α^* be the least ordinal such that $A \cap \alpha^*$ is infinite. Then α^* is a limit ordinal. Pick $\alpha \in A \cap \alpha^*$ with $(\alpha - 1) > \max\{\lambda(\alpha^*, \delta), \lambda(\alpha^*, \delta')\}$. By the second concatenation lemma, then:

$$\begin{aligned} \mathrm{tr}(\alpha, \delta) &= \mathrm{tr}(\alpha^*, \delta)^\frown \mathrm{tr}(\alpha, \alpha^*), & \mathrm{tr}(\alpha - 1, \delta) &= \mathrm{tr}(\alpha^*, \delta)^\frown \mathrm{tr}(\alpha - 1, \alpha^*), \\ \mathrm{tr}(\alpha, \delta') &= \mathrm{tr}(\alpha^*, \delta')^\frown \mathrm{tr}(\alpha, \alpha^*), & \mathrm{tr}(\alpha - 1, \delta') &= \mathrm{tr}(\alpha^*, \delta')^\frown \mathrm{tr}(\alpha - 1, \alpha^*). \end{aligned}$$

Consequently:

$$\begin{aligned} \pi(\rho_{2\delta})(\alpha) &= \rho_{2\delta}(\alpha) - \rho_{2\delta}(\alpha - 1) = \\ &\rho_{2\delta}(\alpha^*) + \rho_{2\alpha^*}(\alpha) - (\rho_{2\delta}(\alpha^*) + \rho_{2\alpha^*}(\alpha - 1)) = \\ &\rho_{2\delta'}(\alpha^*) + \rho_{2\alpha^*}(\alpha) - (\rho_{2\delta'}(\alpha^*) + \rho_{2\alpha^*}(\alpha - 1)) = \\ &\rho_{2\delta'}(\alpha) - \rho_{2\delta'}(\alpha - 1) = \pi(\rho_{2\delta'})(\alpha), \end{aligned}$$

contradicting the fact that $\alpha \in A$. □

To prove that an $\aleph_1$-tree is special, it suffices to prove that this is the case, restricted to some club.

Exercise 3.12. Suppose that $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is an ω_1-tree.

Prove that the following are equivalent:

(1) $\mathcal{T}$ is special;

(2) for some c.u.b. $E \subseteq \omega_1$, $\mathcal{T} \restriction E := \{x \in \mathcal{T} \mid \mathrm{dom}(x) \in E\}$ is the union of countably many antichains;
(3) $\mathcal{T}$ is $\mathbb{Q}$-embeddable.

Next, we establish a sufficient condition for *coherent* $\aleph_1$-trees to be special.

Lemma 3.13 (Peng). *Suppose that $\mathcal{T} \subseteq {}^{<\omega_1}w$ is a coherent $\aleph_1$-tree, for some countable set w.*

Then $\mathcal{T}$ is special iff there exists a c.u.b. $E \subseteq \omega_1$ and a choice $b \in \prod_{\delta \in E} \mathcal{T}_\delta$ whose range is the union of countably many antichains.

Proof. The forward implication is trivial, so we focus on the converse.

Write $V_0 := \emptyset$, and $V_{n+1} := \mathcal{P}(V_n)$ for all $n < \omega$. Then $|V_{n+1}| = 2^{|V_n|}$ is finite, and $V_\omega := \bigcup_{n<\omega} V_n$ is countable. In addition, for any $n < \omega$ and any function $f : n \to V_\omega$, we have $f \in V_\omega$.[2]

We shall want to define a function $c : \mathcal{T} \to V_\omega$ such that the preimage of any singleton is an antichain.

Fix a c.u.b. $E \subseteq \omega_1$, a choice $b \in \prod_{\delta \in E} \mathcal{T}_\delta$, and $d : \mathrm{Im}(b) \to \omega$ such that the preimage of any singleton is an antichain. We may clearly assume that $0 \in E$. For every $\delta < \omega_1$, fix an injection $\psi_\delta : \delta + 1 \to \omega$, and write $\bar\delta := \sup(E \cap \delta)$. Note that $\bar\delta \leq \delta$, and $\bar\delta \in E$, since E is closed.

Next, for all $\delta \in E$ and $x \in \mathcal{T}_\delta$, write b_δ for $b(\delta)$, and consider the finite set $\mathsf{d}_x := \{\alpha < \delta \mid x(\alpha) \neq b_\delta(\alpha)\}$. For all $i < |\mathsf{d}_x|$, let $\mathsf{d}_x(i)$ denote the i^{th} element of d_x.

We now define $c \restriction \mathcal{T}_\delta$ by recursion on $\delta < \omega_1$. Given $x \in \mathcal{T}_\delta$, denote $\bar x := x \restriction \bar\delta$, and let $c(x)$ be the function from $2|\mathsf{d}_{\bar x}|+2$ to V_ω defined by:

$$c(x)(i) := \begin{cases} b_{\bar\delta}(\mathsf{d}_{\bar x}(i)), & \text{if } i < |\mathsf{d}_{\bar x}|; \\ c(\bar x \restriction \mathsf{d}_{\bar x}(j)), & \text{if } i = |\mathsf{d}_{\bar x}| + j < 2|\mathsf{d}_{\bar x}| \\ d(b_{\bar\delta}), & \text{if } i = 2|\mathsf{d}_{\bar x}|; \\ \psi_{\min(E\setminus\delta)}(\delta), & \text{if } i = 2|\mathsf{d}_{\bar x}| + 1. \end{cases}$$

Towards a contradiction, suppose that there exist a pair $x \subsetneq y$ in $\mathcal{T}$ such that $c(x) = c(y)$. Pick such a pair (x, y) with $(\mathrm{dom}(x), \mathrm{dom}(y))$ of lexicographically least possible value, say, (δ, δ'). In particular, $\delta < \delta'$.[3]

Write $n := |\mathsf{d}_{\bar x}|$, so that $c(x) = c(y)$ is a function from $2n + 2$ to V_ω.

Claim 3.13.1. *$\bar\delta < \bar\delta'$;*

Proof. Otherwise, $\bar\delta = \bar\delta'$, That is, $\sup(E \cap \delta) = \sup(E \cap \delta')$. But, then $\min(E \setminus \delta) = \min(E \setminus \delta')$ and so by $\delta < \delta'$, we should have had

[2] In fact, $(V_\omega, \in)$ is a model of ZFC minus the axiom of infinity.

[3] Note that x is a proper subsequence of y.

$\psi_{\min(E\setminus\delta)}(\delta) \neq \psi_{\min(E\setminus\delta')}(\delta')$, contradicting the fact that $c(x)(2n+1) = c(y)(2n+1)$. $\square$

To be continued next week... $\square$

We continue with the proof of the Lemma from last week.

Lemma 4.1 (Peng)**.** *Suppose that $\mathcal{T} \subseteq {}^{<\omega_1}w$ is a coherent $\aleph_1$-tree, for some countable set w. Suppose that $d : \{b_\delta \mid \delta \in E\} \rightarrow \omega$ is a function, where:*

- *E is a c.u.b in ω_1;*
- *for all $\delta \in E$, $b_\delta \in \mathcal{T}_\delta$, and*
- *for all $m < \omega$, $d^{-1}\{m\}$ is an antihcain.*

Then $\mathcal{T}$ is special.

Proof. We may assume that $0 \in E$. For every $\delta < \omega_1$, fix an injection $\psi_\delta : \delta + 1 \rightarrow \omega$, and write $\bar{\delta} := \sup(E \cap \delta)$, so that $\bar{\delta} \in E \cap (\delta + 1)$

For all $\delta \in E$ and $x \in \mathcal{T}_\delta$, we compare x with b_δ, letting $\mathsf{d}_x := \{\alpha < \delta \mid x(\alpha) \neq b_\delta(\alpha)\}$. For all $i < |\mathsf{d}_x|$, let $\mathsf{d}_x(i)$ denote the i^{th} element of d_x.

Let V_ω denote the collection of all hereditary finite sets. We define a function $c : \mathcal{T} \rightarrow V_\omega$, as follows. Define $c \restriction \mathcal{T}_\delta$ by recursion on $\delta < \omega_1$. Given $x \in \mathcal{T}_\delta$, denote $\bar{x} := x \restriction \bar{\delta}$, and let $c(x)$ be the function from $2|\mathsf{d}_{\bar{x}}| + 2$ to V_ω defined by:

$$c(x)(i) := \begin{cases} b_{\bar{\delta}}(\mathsf{d}_{\bar{x}}(i)), & \text{if } i < |\mathsf{d}_{\bar{x}}|; \\ c(\bar{x} \restriction \mathsf{d}_{\bar{x}}(j)), & \text{if } i = |\mathsf{d}_{\bar{x}}| + j < 2|\mathsf{d}_{\bar{x}}| \\ d(b_{\bar{\delta}}), & \text{if } i = 2|\mathsf{d}_{\bar{x}}|; \\ \psi_{\min(E \setminus \delta)}(\delta), & \text{if } i = 2|\mathsf{d}_{\bar{x}}| + 1. \end{cases}$$

Towards a contradiction, suppose that there exist a pair $x \subsetneq y$ in $\mathcal{T}$ such that $c(x) = c(y)$. Pick such a pair (x, y) with $(\mathrm{dom}(x), \mathrm{dom}(y))$ of lexicographically least possible value, say, (δ, δ'). Last week we have noticed that $\delta < \delta'$ and furthermore $\bar{\delta} < \bar{\delta}'$.

Write $n := |\mathsf{d}_{\bar{x}}|$, so that $c(x) = c(y)$ is a function from $2n + 2$ to V_ω.

Claim 4.1.1. *There exists $j < n$ such that $\bar{x} \restriction \mathsf{d}_{\bar{x}}(j) \subsetneq \bar{y} \restriction \mathsf{d}_{\bar{y}}(j)$ or $\bar{y} \restriction \mathsf{d}_{\bar{y}}(j) \subsetneq \bar{x} \restriction \mathsf{d}_{\bar{x}}(j)$.*[1]

Proof. We have $\bar{\delta} < \bar{\delta}'$, so that $b_{\bar{\delta}} \neq b_{\bar{\delta}'}$. But $d(b_{\bar{\delta}}) = c(x)(2n) = c(y)(2n) = d(b_{\bar{\delta}'})$, and hence $b_{\bar{\delta}}$ must be incompatible with $b_{\bar{\delta}'}$. Let $\gamma < \bar{\delta}$ be the least such that $b_{\bar{\delta}}(\gamma) \neq b_{\bar{\delta}'}(\gamma)$. As $x \subseteq y$, we have $\bar{x} \restriction (\gamma + 1) = \bar{y} \restriction (\gamma + 1)$, and hence, for all $\alpha < \gamma$,

$$\bar{x}(\alpha) = b_{\bar{\delta}}(\alpha) \text{ iff } \bar{x}(\alpha) = b_{\bar{\delta}'}(\alpha) \text{ iff } \bar{y}(\alpha) = b_{\bar{\delta}'}(\alpha).$$

So $\mathsf{d}_{\bar{x}} \cap \gamma = \mathsf{d}_{\bar{y}} \cap \gamma$. We claim that $j := |\mathsf{d}_{\bar{x}} \cap \gamma|$ is as sought.

[1] In particular, $n > 0$.

▶ If $\bar{x}(\gamma) = b_{\bar{\delta}}(\gamma)$, then $\bar{x}(\gamma) \neq b_{\bar{\delta}'}(\gamma)$, and as $x \restriction (\gamma+1) = y \restriction (\gamma+1)$, we infer that $\bar{y}(\gamma) \neq b_{\bar{\delta}'}(\gamma)$. So, in this case, $\gamma \in \mathsf{d}_{\bar{y}} \setminus \mathsf{d}_{\bar{x}}$ and $\mathsf{d}_{\bar{y}}(j) = \gamma < \mathsf{d}_{\bar{x}}(j)$. Consequently, $\bar{y} \restriction \mathsf{d}_{\bar{y}}(j) \subset \bar{x} \restriction \mathsf{d}_{\bar{x}}(j)$.

▶ If $\bar{x}(\gamma) \neq b_{\bar{\delta}}(\gamma)$, then $\gamma \in \mathsf{d}_{\bar{x}}$, so that $\mathsf{d}_{\bar{x}}(j) = \gamma$. Now, if $\mathsf{d}_{\bar{y}}(j)$ were to equal γ, as well, then $b_{\bar{\delta}}(\gamma) = c(x)(j) = c(y)(j) = b_{\bar{\delta}'}(\gamma)$, contradicting the choice of γ. So, in this case, $\bar{x} \restriction \mathsf{d}_{\bar{x}}(j) \subset \bar{y} \restriction \mathsf{d}_{\bar{y}}(j)$. □

Let $j < n$ be given by the preceding. Then $\bar{x} \restriction \mathsf{d}_{\bar{x}}(j)$ and $\bar{y} \restriction \mathsf{d}_{\bar{y}}(j)$ are two distinct compatible elements, and

$$c(\bar{x} \restriction \mathsf{d}_{\bar{x}}(j)) = c(x)(n+j) = c(y)(n+j) = c(\bar{y} \restriction \mathsf{d}_{\bar{y}}(j)).$$

But then, the pair $(\mathsf{d}_{\bar{x}}(j), \mathsf{d}_{\bar{y}}(j)) \in \bar{\delta} \times \bar{\delta}'$ contradicts the minimality of the pair (δ, δ'). □

Corollary 4.2 (Peng)**.** *$\mathcal{T}(\rho_2)$ is special.*

Proof. Last week we have seen that $\{\rho_{2\delta} \mid \delta < \omega_1\}$ is the countable union of antichains. Since π is an isomorphism, so is $\{\pi(\rho_{2\delta}) \mid \delta < \omega_1\}$. By Lemma 4.1, then, $\mathcal{T}_\pi(\rho_2)$ is special. Consequently, $\mathcal{T}(\rho_2)$ is special. □

Exercise 4.3. *Last week, before proving that $\mathcal{T}(\rho_2)$ is isomorphic to $\mathcal{T}_\pi(\rho_2)$, we have added the hypothesis that 0 is in the ladder C_δ for every nonzero $\delta < \omega_1$. Prove that $\mathcal{T}(\rho_2)$ is special regardless of this assumption.*

Exercise 4.4. *Prove that every Aronszjan tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega_1$ is order-isomorphic to a subtree of ${}^{<\omega_1}\omega$.*

Observation 4.5. *If there exists a nonspecial Aronszajn tree, then there exists an Aronszajn tree that is not isomorphic to a coherent one.*

Proof. Suppose that there exists a nonspecial Aronszajn tree. In particular, we may pick $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ which is Aronszajn and nonspecial. Also, pick $\mathcal{S} \subseteq {}^{<\omega_1}\omega$ which is Aronszajn and special, e.g., $\mathcal{S} = \mathcal{T}(\rho_2)$.

Clearly, $\mathcal{T} \cup \mathcal{S}$ is an ω_1-tree. By the pigeonhole principle, it is moreover Aronszajn.

Now, if $\mathcal{T} \cup \mathcal{S}$ was isomorphic to a coherent tree $\mathcal{C}$, then by Lemma 4.1, the copy of the special tree $\mathcal{S}$ in $\mathcal{C}$ would imply that $\mathcal{C}$ is special. However, by $\mathcal{T} \subseteq \mathcal{T} \cup \mathcal{S}$, the tree $\mathcal{T} \cup S$ cannot be special, and so $\mathcal{C}$ must be nonspecial. □

PFA directly implies its version for ccc forcings, Martin's axiom. In cardinal arithmetic, PFA implies 2^{\aleph _{0}}=\aleph _{2}}

The proper forcing axiom (PFA) implies that any two Aronszajn trees $\mathcal{T}, \mathcal{T}' \subseteq {}^{<\omega_1}\omega_1$ are *club-isomorphic*. That is, for some c.u.b. $E \subseteq \omega_1$, $(\mathcal{T} \restriction E, \subset)$, and $(\mathcal{T}' \restriction E, \subset)$ are order-isomorphic.

Problem 4.6. *Is every special Aronszajn tree club-isomorphic to a coherent one?*

Exercise 4.7. *Suppose that $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is an Aronszajn tree.*

Prove that the following are equivalent:

1. *$\mathcal{T}$ is $\mathbb{R}$-embeddable;*
2. *For some c.u.b. $E \subseteq \omega_1$, $\mathcal{T} \restriction \{\alpha + 1 \mid \alpha \in E\}$ is the countable union of antichains;*
3. *For any nonstationary $A \subseteq \omega_1$, $\mathcal{T} \restriction A$ is the countable union of antichains.*

Definition 4.8 (Maximal weight)**.** Define $\rho_1 : [\omega_1]^2 \to \omega$ by letting for all $\beta < \delta < \omega_1$:

$$\boxed{\rho_1(\beta,\delta) := \max\{|C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta| \mid i < \rho_2(\beta,\delta)\}.}$$ [2]

max # of ordinals in the ladder below β.

For every $\delta < \omega_1$, derive the fiber map $\rho_{1\delta} : \delta \to \omega$ via $\rho_{1\delta}(\beta) := \rho_1(\beta,\delta)$. For simplicity, we do consider $\rho_{1\delta}(\beta)$ for $\beta = \delta$, setting $\rho_{1\delta}(\delta) := 0$. For all $n < \omega$, write

- $\boxed{D_n(\delta) := \{\alpha < \delta \mid \rho_{1\delta}(\alpha) = n\},}$ and
- $\boxed{D_{\leq n}(\delta) := \{\alpha < \delta \mid \rho_{1\delta}(\alpha) \leq n\}}$.

Lemma 4.9. *For every $\delta < \omega_1$, $\rho_{1\delta}$ is finite-to-one.*

Equivalently, $D_n(\delta)$ and $D_{\leq n}(\delta)$ are finite for all $\delta < \omega_1$ and $n < \omega$.

Proof. Suppose not, and let $\delta < \omega_1$ be the least counterexample.

Pick $n < \omega$ for which $D_{\leq n}(\delta)$ infinite. As every $\alpha \in D_{\leq n}(\delta)$ satisfies $|C_\delta \cap \alpha| \leq \rho_{1\delta}(\alpha) \leq n$, we may define $f : D_{\leq n}(\delta) \to n+1$ by stipulating

$$f(\alpha) := |C_\delta \cap \alpha|.$$

Since we assume that $D_{\leq n}(\delta)$ is infinite, let us fix a subset $X \subseteq D_{\leq n}(\delta)$ of order-type ω on which f is constant. In particular, $\min(C_\delta \setminus \alpha_1) = \min(C_\delta \setminus \alpha_2)$ for all $\alpha_1, \alpha_2 \in X$, say it is δ'. Then $X \in [\delta']^\omega$, and so by $\delta' < \delta$ and minimality of the latter, we may find some $\alpha' \in X$ such that $\rho_{1\delta'}(\alpha') > n$. That is, $\alpha' \in X \setminus D_{\leq n}(\delta')$. By $\min(C_\delta \setminus \alpha') = \delta'$, we have $\mathrm{tr}(\alpha',\delta) = \langle \delta \rangle ^\frown \mathrm{tr}(\alpha',\delta')$, and hence

C_δ has same number of ordinals below α_1 and α_2, hence the same for above (it doesn't pass anyone). Hence the min is the same.

$$\rho_{1\delta}(\alpha') = \max\{|C_\delta \cap \alpha'|, \rho_{1\delta'}(\alpha')\} > n,$$

contradicting the fact that $\alpha' \in X \subseteq D_{\leq n}(\delta)$. □

Consider the tree $\mathcal{T}(\rho_1) := \{\rho_{1\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$.

Lemma 4.10. *$\mathcal{T}(\rho_1)$ is coherent.*

[2] This is indeed quite similar to the function $\lambda(\beta,\delta) := \max\{\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) \mid i < \rho_2(\beta,\delta)\}$ from Lecture #2.

Proof. Suppose not. Then there exist a pair of countable ordinals $\gamma < \delta$ for which $\{\alpha < \gamma \mid \rho_{1\gamma}(\alpha) \neq \rho_{1\delta}(\alpha)\}$ is infinite. Let $\delta < \omega_1$ be the least for which there exist $\gamma < \delta$ and a subset $X \subseteq \gamma$ of order-type ω with $\rho_{1\gamma}(\alpha) \neq \rho_{1\delta}(\alpha)$ for all $\alpha \in X$. Put $\beta := \sup(X)$, $\beta^+ := \min(C_\delta \setminus \beta)$ and $\beta^- := \sup(C_\delta \cap \beta)$. By $\beta \leq \gamma < \delta$, $C_\delta \cap \beta$ is finite, and so $\mathrm{cf}(\beta) = \omega$ entails

$$\beta^- < \beta \leq \beta^+ < \delta.$$

Put $n := |C_\delta \cap \beta|$, and then $X' := \{\alpha \in X \setminus (\beta^- + 1) \mid \rho_{1\delta}(\alpha) > n\}$. By Lemma 4.9, we have $\mathrm{otp}(X') = \omega$, so that $\sup(X') = \sup(X) = \beta$. By minimality of δ, we get from $\beta \leq \max\{\gamma, \beta^+\} < \delta$ that there exists $\alpha \in X'$ such that $\rho_{1\gamma}(\alpha) = \rho_{1\beta^+}(\alpha)$.

By $\beta^- < \alpha < \beta \leq \beta^+$, we have:

(1) $\min(C_\delta \setminus \alpha) = \min(C_\delta \setminus \beta) = \beta^+$, so that $\mathrm{tr}(\alpha, \delta) = \langle \delta \rangle^\frown \mathrm{tr}(\alpha, \beta^+)$;
(2) $|C_\delta \cap \alpha| = |C_\delta \cap \beta| = n < \rho_{1\delta}(\alpha)$, so that $\rho_{1\delta}(\alpha) = \max\{|C_\delta \cap \alpha|, \rho_{1\beta^+}(\alpha)\} = \rho_{1\beta^+}(\alpha)$.

It follows that $\rho_{1\delta}(\alpha) = \rho_{1\beta^+}(\alpha) = \rho_{1\gamma}(\alpha)$, contradicting the fact that $\alpha \in X$. □

Corollary 4.11. *$\mathcal{T}(\rho_1)$ is a coherent Aronszajn tree.*

Proof. By Lemma 4.10, for every $\beta < \omega_1$, any element of the β^{th}-level of $\mathcal{T}(\rho_1)$ is a function from β to ω that differs from $\rho_{1\beta}$ on a finite set, and hence all levels of $\mathcal{T}(\rho_1)$ are countable.

Finally, if $b : \omega_1 \to \omega$ is a cofinal branch through $\mathcal{T}(\rho_1)$, then by the pigeonhole principle, we may fix an uncountable $X \subseteq \omega_1$ on which b is constant, with value, say, k. Pick $\beta < \omega_1$ such that $X \cap \beta$ is infinite. By $b \restriction \beta \in \mathcal{T}(\rho_1)$, let us pick $\delta < \omega_1$ such that $b \restriction \beta = \rho_{1\delta} \restriction \beta$. Then $D_k(\delta)$ covers the infinite set $X \cap \beta$, contradicting Lemma 4.9. □

forcing is a poset. forcing conditions are the elements of the poset. we say p is stronger than q if p<q.

Definition 4.12. An ω_1-tree $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is said to be *normal* if for all $x \in \mathcal{T}$ and $\beta < \omega_1$, there exists some $y \in \mathcal{T}_\beta$ which is compatible with x.

Exercise 4.13. *Prove that for all $\gamma < \delta < \omega_1$, if $\gamma = \min(C_\delta)$, then $\rho_{1\gamma} \subseteq \rho_{1\delta}$. Conclude that for an appropriate choice of a ladder system (in ZFC), $\mathcal{T}(\rho_1)$ is moreover normal.*

Exercise 4.14. *Prove that any normal Aronszajn tree is splitting. That is, for any node x in such a tree, there are nodes y, z extending it such that y and z are incompatible.*

Coming back to Corollary 4.11, it is also possible to prove that $\mathcal{T}(\rho_1)$ does not admit a cofinal branch by showing that ρ_1 enjoys an unboundedness feature similar to that of ρ_2.

Exercise 4.15. *Prove that ρ_1 witnesses that $\mathrm{U}(\omega_1, \omega, \mathit{finite})$ holds.*

Definition 5.1. For two sets of ordinals A and B, write $A \sqsubseteq B$ iff there exists some β such that $A = B \cap \beta$. We write $A \sqsubset B$ whenever $A \sqsubseteq B$ and $A \neq B$.

Recall the definition of the *maximal weight* function:

Definition 5.2. Define $\rho_1 : [\omega_1]^2 \to \omega$ by letting for all $\beta < \delta < \omega_1$:

$$\rho_1(\beta, \delta) := \max\{|C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta| \mid i < \rho_2(\beta, \delta)\}.^1$$

For every $\delta < \omega_1$, we derive the fiber map $\rho_{1\delta} : \delta \to \omega$ in the usual way. By convention, we set $\rho_{1\delta}(\delta) := 0$. For all $n < \omega$, write

- $D_n(\delta) := \{\alpha < \delta \mid \rho_{1\delta}(\alpha) = n\}$, and
- $D_{\leq n}(\delta) := \{\alpha < \delta \mid \rho_{1\delta}(\alpha) \leq n\}$.

We proved that, for every $\delta < \omega_1$ and $n < \omega$, $D_{\leq n}(\delta)$ is finite. Utilizing this, we can now introduce an injective variation of ρ_1.

Definition 5.3. Define $\bar{\rho}_1 : [\omega_1]^2 \to \omega \times \omega$ by letting for all $\beta < \delta < \omega_1$:

$$\bar{\rho}_1(\beta, \delta) := (\rho_{1\delta}(\beta), |D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta|).$$

That is, $\bar{\rho}_1(\beta, \delta) = (\rho_{1\delta}(\beta), |\{\alpha < \beta \mid \rho_{1\delta}(\alpha) = \rho_{1\delta}(\beta)\}|)$.

Lemma 5.4. *For all $\delta < \omega_1$, the fiber map $\bar{\rho}_{1\delta} : \delta \to \omega \times \omega$ is injective.*

Proof. Let $\alpha < \beta < \delta$ be arbitrary. If $\rho_{1\delta}(\alpha) \neq \rho_{1\delta}(\beta)$, then clearly $\bar{\rho}_{1\delta}(\alpha) \neq \bar{\rho}_{1\delta}(\beta)$. Now, suppose that $\rho_{1\delta}(\alpha) = \rho_{1\delta}(\beta)$, say it is equal to n. Then:

- $(D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta) \cap \alpha = D_n(\delta) \cap \alpha = D_{\rho_{1\delta}(\alpha)}(\delta) \cap \alpha$, and
- $\alpha \in D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta$.

So $D_{\rho_{1\delta}(\beta)}(\delta) \cap \alpha \sqsubset D_{\rho_{1\delta}(\beta)}(\delta) \cap \beta$, and since the latter are finite sets, their cardinality is not the same. Consequently, $\bar{\rho}_{1\delta}(\alpha) \neq \bar{\rho}_{1\delta}(\beta)$. $\square$

We have also maintained coherence:

Lemma 5.5. *For all $\gamma < \delta < \omega_1$, the set $\{\beta < \gamma \mid \bar{\rho}_{1\gamma}(\beta) \neq \bar{\rho}_{1\delta}(\beta)\}$ is finite.*

Proof. Let $\gamma < \delta$ be an arbitrary pair of countable ordinals. We need to show that the set

$$X := \{\beta < \gamma \mid \bar{\rho}_{1\gamma}(\beta) \neq \bar{\rho}_{1\delta}(\beta)\}$$

is finite. Let:

- $A := \{\alpha < \gamma \mid \rho_{1\gamma}(\alpha) \neq \rho_{1\delta}(\alpha)\}$,
- $n := \max\{0, \rho_{1\gamma}(\alpha), \rho_{1\delta}(\alpha) \mid \alpha \in A\}$, and

[1]This is indeed quite similar to the function $\lambda(\beta, \delta) := \max\{\sup(C_{\mathrm{Tr}(\beta,\delta)(i)} \cap \beta) \mid i < \rho_2(\beta, \delta)\}$ from Lecture #2.

- $B := D_{\leq n}(\gamma) \cup D_{\leq n}(\delta)$.

Clearly, A, n, and B are finite, with $A \subseteq B$. We claim that $X \subseteq B$. To see this, let $\beta \in \gamma \setminus B$ be arbitrary.

▶ By $A \subseteq B$, we have $\beta \notin A$, and hence $\rho_{1\gamma}(\beta) = \rho_{1\delta}(\beta)$. So, the left coordinate of $\bar{\rho}_{1\gamma}(\beta)$ is equal to that of $\bar{\rho}_{1\delta}(\beta)$.

▶ By $\beta \notin B$, there exists an integer $m > n$ such that $\rho_{1\gamma}(\beta) = \rho_{1\delta}(\beta) = m$, and hence we need to show that $D_m(\gamma) \cap \beta = D_m(\delta) \cap \beta$. But, in fact, we moreover have $D_m(\gamma) = D_m(\delta) \cap \gamma$, simply because $D_m(\gamma) \cap A = \emptyset = D_m(\delta) \cap A$ as a consequence of $m > n$. □

Corollary 5.6. *$\mathcal{T}(\bar{\rho}_1) := \{\bar{\rho}_{1\delta} \restriction \beta \mid \beta \leq \delta < \omega_1\}$ is a coherent Aronszajn tree consisting of injections!*

Exercise 5.7. *Show that $\mathcal{T}(\rho_1)$ is special iff $\mathcal{T}(\bar{\rho}_1)$ is special.*

Recall that a *Souslin tree* is an Aronszajn tree that does not admit an uncountable antichain. In a paper from 1984, Shelah proved that adding a Cohen real introduces a Souslin tree. A relatively simple proof of this fact goes as follows.

Exercise 5.8 (Todorcevic, 1987). *If $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is a coherent Aronszajn tree consisting of injections, and $c : \omega \to \omega$ is a Cohen real, then $\{c \circ t \mid t \in \mathcal{T}\}$ is a coherent Souslin tree.*

Exercise 5.9. *Prove that Souslin trees are not $\mathbb{R}$-embeddable.*

Lemma 5.10. *$\mathcal{T}(\rho_1)$ and $\mathcal{T}(\bar{\rho}_1)$ are $\mathbb{R}$-embeddable.*

Proof. First, define a function $f : \mathcal{T}(\rho_1) \to \mathcal{T}(\bar{\rho}_1)$, as follows. Given a finite-to-one function $t : \gamma \to \omega$, define $f(t) : \gamma \to \omega \times \omega$ by letting for all $\beta < \gamma$:

$$f(t)(\beta) := (t(\beta), |\{\alpha < \beta \mid t(\alpha) = t(\beta)|).$$

Evidently, f witnesses that $(\mathcal{T}(\rho_1), \subset)$ and $(\mathcal{T}(\bar{\rho}_1), \subset)$ are order-isomorphic. Next, define $g : \mathcal{T}(\bar{\rho}_1) \to \mathcal{P}(\omega)$ by stipulating $g(t) := \{2^a 3^b \mid (a, b) \in \mathrm{Im}(t)\}$. As elements of $\mathcal{T}(\bar{\rho}_1)$ are injections, we get that for any pair $s \subset t$ of elements of $\mathcal{T}(\bar{\rho}_1)$, $g(s) \subset g(t)$.

Finally, define $h : \mathcal{P}(\omega) \to [0, 1]$ by stipulating $g(A) := \sum_{n \in A} \frac{1}{2^{n+1}}$. Evidently, for all $A \subset B \subseteq \omega$, we have

$$g(B) = g(A) + g(B \setminus A) > g(A),$$

so that h is an order-preserving map from $(\mathcal{P}(\omega), \subset)$ to the real line $(\mathbb{R}, <)$. It follows that $h \circ g \circ f$ witnesses that $\mathcal{T}(\rho_1)$ is $\mathbb{R}$-embeddable, and $h \circ g$ witnesses that $\mathcal{T}(\bar{\rho}_1)$ is $\mathbb{R}$-embeddable. □

Is $\mathcal{T}(\rho_1)$ is special? This is surely, consistently, the case, as it is consistent with ZFC that *all* Aronszajn trees are special. However, the following is Question 2.2.18 from Stevo's book.

Problem 5.11 (Todorcevic)**.** *What is the condition one needs to put on a given ladder system in order to guarantee that the corresponding tree $\mathcal{T}(\rho_1)$ be special?*

Two related problems read as follows.

Problem 5.12. *Must there exist a ladder system for which the corresponding tree $\mathcal{T}(\rho_1)$ be special?*

Problem 5.13. *Suppose that there exists a nonspecial Aronszajn tree. Must there exist a ladder system for which the corresponding tree $\mathcal{T}(\rho_1)$ be nonspecial?*[2]

We shall soon show that it is consistent that for some choice of a ladder system, the corresponding tree $\mathcal{T}(\rho_1)$ be nonspecial. This will give a consistent example of an Aronszajn tree which is $\mathbb{R}$-embeddable but not $\mathbb{Q}$-embeddable.

Cconsider the following principle.

Definition 5.14 (Hrušák and Martínez-Ranero)**.** $\star_0$ asserts the existence of a ladder system $\langle C_\delta \mid \delta < \omega_1\rangle$ such that:

- for every $\delta < \omega_1$, $C_{\delta+1} = \{\delta\}$;
- for every limit $\delta < \omega_1$, C_δ is some cofinal subset of δ of order-type ω;
- for every function $f : \omega_1 \to \omega$, there exist a pair of counatble limit ordinals $\gamma < \delta$ with $f(\gamma) = f(\delta)$ such that $\gamma \in C_\delta$ and $C_\delta \cap \gamma \sqsubset C_\gamma$.

Exercise 5.15 (Hrušák and Martínez-Ranero)**.** *Show that $\star_0$ holds after forcing to add a Cohen real.*

Conclude that $\star_0$ is consistent with Martin's Axiom for σ-centered posets.

The next theorem implies, in particular, that $\star_0$ is inconsistent with Martin's Axiom.

Theorem 5.16 (Hrušák and Martínez-Ranero, 2005)**.** *Suppose that ρ_1 is derived from a walk along a $\star_0$-sequence. Then $\mathcal{T}(\rho_1)$ is nonspecial.*

Proof. Suppose that $\{A_n \mid n < \omega\}$ is a cover of $\mathcal{T}(\rho_1)$ by antichains. Define $f : \omega_1 \to \omega$ by stipulating $f(\delta) := \min\{n < \omega \mid \rho_{1\delta} \in A_n\}$. Now, pick a pair of limit ordinals $\gamma < \delta$ with $f(\gamma) = f(\delta)$ such that $\gamma \in C_\delta$ and $C_\delta \cap \gamma \sqsubset C_\gamma$. We shall obtain a contradiction by showing that $\rho_{1\gamma}, \rho_{1\delta}$ (that belong to the same antichain) are actually compatible.

[2]Recall that by the definition given in Lecture #1, all ladders are required to be of order-type $\leq \omega$.

Let $\alpha < \gamma$ be arbitrary. We shall show that $\rho_{1\gamma}(\alpha) = \rho_{1\delta}(\alpha)$.

Let $\{\delta_i \mid i < \omega\}$ denote the increasing enumeration of C_δ, and $\{\gamma_i \mid i < \omega\}$ denote the increasing enumeration of C_γ. Let $n = |C_\delta \cap \gamma|$. By $C_\delta \cap \gamma \sqsubseteq C_\gamma$, we have $\delta_i = \gamma_i$ for all $i < n$, and $\delta_n = \gamma$. There are two cases to consider:

- ▶ There is $i < n$ such that $\alpha \leq \delta_i$. Then for the least such i, we have
  - $\mathrm{tr}(\alpha, \delta) = \langle \delta \rangle ^\frown \mathrm{tr}(\alpha, \delta_i)$,
  - $\mathrm{tr}(\alpha, \gamma) = \langle \gamma \rangle ^\frown \mathrm{tr}(\alpha, \delta_i)$,

  and
  $$\rho_{1\gamma}(\alpha) = \max\{|C_\gamma \cap \alpha|, \rho_{1\delta_i}(\alpha)\} =$$
  $$\max\{|C_\delta \cap \gamma \cap \alpha|, \rho_{1\delta_i}(\alpha)\} =$$
  $$\max\{|C_\delta \cap \alpha|, \rho_{1\delta_i}(\alpha)\} = \rho_{1\delta}(\alpha).$$
- ▶ There is no $i < n$ such that $\alpha < \delta_i$. Then, by $n = |C_\delta \cap \gamma|$, this means that $\min(C_\delta \setminus \alpha) = \delta_n = \gamma$. Consequently, $\mathrm{tr}(\alpha, \delta) = \langle \delta \rangle ^\frown \mathrm{tr}(\alpha, \gamma)$, so that $\rho_{1\delta}(\alpha) = \max\{|C_\delta \cap \alpha|, \rho_{1\gamma}(\alpha)\}$. By definition, we have $\rho_{1\gamma}(\alpha) \geq |C_\gamma \cap \alpha|$. As $C_\gamma \supseteq C_\delta \cap \gamma$, we have $|C_\gamma \cap \alpha| \geq |C_\delta \cap \gamma \cap \alpha| = |C_\delta \cap \alpha|$, and then $\rho_{1\delta}(\alpha) = \rho_{1\gamma}(\alpha)$.

□

Definition 5.17. The principle $\diamondsuit$ asserts the existence of a sequence $\langle X_\alpha \mid \alpha < \omega_1 \rangle$ such that for every $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 \mid X_\alpha = X \cap \alpha\}$ is stationary.

Exercise 5.18. *Show that $\diamondsuit$ implies that $2^{\aleph_0} = \aleph_1$.*

Exercise 5.19 (P. Larson)**.** *Show that $\diamondsuit$ entails $\star_0$.*

Corollary 5.20. *$\diamondsuit$ entails the existence of an ω_1-tree which is $\mathbb{R}$-embeddable, but not $\mathbb{Q}$-embeddable.*

Definition 5.21. The principle $\clubsuit$ asserts the existence of a ladder system $\langle C_\alpha \mid \alpha < \omega_1 \rangle$ such that for every uncountable $X \subseteq \omega_1$, there exists a nonzero limit ordinal $\alpha < \omega_1$ such that $C_\alpha \subseteq X$.

Exercise 5.22 (Devlin)**.** *Show that $\diamondsuit$ holds iff ($2^{\aleph_0} = \aleph_1$ and $\clubsuit$ holds).*

Problem 5.23. *Does $\clubsuit$ entail $\star_0$?*

Is $\clubsuit$ consistent with the assertion that all Aronszajn trees are special?

Exercise 5.24. *Show that all of the following are equivalent:*

1. *$\diamondsuit$;*
2. *there exists sequence $\langle (X_\alpha, Y_\alpha) \mid \alpha < \omega_1 \rangle$ such that, for all $X \subseteq \omega_1$ and $Y \subseteq \omega_1$, the set $\{\alpha < \omega_1 \mid X_\alpha = X \cap \alpha \;\&\; Y_\alpha = Y \cap \alpha\}$ is stationary;*

(3) *there exists sequence* $\langle X_\alpha \mid \alpha < \omega_1\rangle$ *and a function* $f : \omega_1 \to \omega_1$ *such that, for all* $X \subseteq \omega_1$ *and* $i < \omega_1$*, the set* $\{\alpha < \omega_1 \mid X_\alpha = X \cap \alpha \ \&\ f(\alpha) = i\}$ *is stationary;*
(4) *there exists a sequence* $\langle g_\alpha \mid \alpha < \omega_1\rangle$ *such that, for every function* $g : \omega_1 \to \omega_1$*, the set* $\{\alpha < \omega_1 \mid g_\alpha = g \restriction \alpha\}$ *is stationary;*
(5) *there exists a matrix* $\langle X_{\alpha,n} \mid \alpha < \omega_1, n < \omega\rangle$ *such that for every* $X \subseteq \omega_1$*, there exists a nonzero limit ordinal* $\alpha < \omega_1$ *and some integer* $n < \omega$ *such that* $X_{\alpha,n} = X \cap \alpha$*.*

Now let us turn to study abstract forms of ϱ functions.

Definition 5.25. For infinite cardinals λ, χ, a function $\varrho : [\lambda^+]^2 \to \lambda$ is said to be:[3]

- *locally small*, if $|\{\alpha < \beta \mid \varrho_\beta(\alpha) \leq \nu\}| < \lambda$ for all $\nu < \lambda$ and $\beta < \lambda^+$;
- *χ-coherent*, if $|\{\alpha < \beta \mid \varrho_\beta(\alpha) \neq \varrho_\gamma(\alpha)\}| < \chi$ for all $\beta < \gamma < \lambda^+$.

We saw earlier that $\rho_1 : [\omega_1]^2 \to \omega$ is locally small and ω-coherent.

Theorem 5.26 (B. König, 2003)**.** *There exists no* $\varrho : [\omega_2]^2 \to \omega_1$ *which is locally small and* ω*-coherent.*

Proof. Towards a contradiction, suppose that $\varrho : [\omega_2]^2 \to \omega_1$ is a counterexample. Since ϱ is locally small, we get that for every $\nu < \omega_1$, $D_{\leq\nu}(\delta) := \{\alpha < \delta \mid \varrho_\delta(\alpha) \leq \nu\}$ is countable.

For every ordinal $\delta < \omega_2$, let us denote $\delta^* := \delta + \omega_1$.

Claim 5.26.1. *For every* $\delta < \omega_2$*, there exists some* $\nu < \omega_1$ *such that:*

$$\mathrm{otp}(D_{\leq\nu}(\delta)) + \omega < \mathrm{otp}(D_{\leq\nu}(\delta^*)).$$

Proof. Since $\mathsf{d}(\delta, \delta^*) := \{\alpha < \delta \mid \varrho_\delta(\alpha) \neq \varrho_{\delta^*}(\alpha)\}$ is countable (in fact, finite), $\nu_0 := \sup\{\varrho_\delta(\alpha), \varrho_{\delta^*}(\alpha) \mid \alpha \in \mathsf{d}(\delta, \delta^*)\}$ is a countable ordinal.

Since the ordinal-interval $[\delta, \delta + \omega + 1]$ is countable, we have that $\nu_1 := \sup(\varrho_{\delta^*}``[\delta, \delta + \omega + 1])$ is yet another countable ordinal.

Put $\nu := \max\{\nu_0, \nu_1\} + 1$. Then:

▶ From $\nu > \nu_0$, we have $D_{\leq\nu}(\delta) = D_{\leq\nu}(\delta^*) \cap \delta$.

▶ From $\nu > \nu_1$, we have $[\delta, \delta + \omega + 1] \subseteq D_{\leq\nu}(\delta^*) \cap [\delta, \delta^*)$, so that $\omega < \mathrm{otp}(D_{\leq\nu}(\delta^*) \cap [\delta, \delta^*))$.

Altogether, $\mathrm{otp}(D_{\leq\nu}(\delta)) + \omega < \mathrm{otp}(D_{\leq\nu}(\delta^*))$. □

Consider the set $E^{\omega_2}_{\omega_1} := \{\delta < \omega_2 \mid \mathrm{cf}(\delta) = \omega_1\}$. For each $\delta \in E^{\omega_1}_{\omega_1}$, let $\nu(\delta)$ be given by the preceding claim. By the pigeonhole principle, let

[3] As usual, $\varrho_\delta : \delta \to \lambda$ denotes the fiber map satisfying $\varrho_\delta(\alpha) = \varrho(\alpha, \delta)$ for all $\alpha < \delta$.

us fix some $\nu < \omega_1$ along with $A \subseteq E^{\omega_2}_{\omega_1}$ of size $\aleph_2$ for which $\nu(\delta) = \nu$ for all $\delta \in A$.

The next claim is really where we use the fact that ϱ is ω-coherent.

Claim 5.26.2. *For every $\gamma \leq \delta < \omega_2$:*

$$\mathrm{otp}(D_{\leq\nu}(\gamma)) < \mathrm{otp}(D_{\leq\nu}(\delta)) + \omega.$$

Proof. Since $\mathsf{d}(\gamma,\delta) := \{\alpha < \gamma \mid \varrho_\gamma(\alpha) \neq \varrho_\delta(\alpha)\}$ is a finite set, there exists some integer $z \in \mathbb{Z}$ such that

$$\mathrm{otp}(D_{\leq\nu}(\gamma)) = \mathrm{otp}(D_{\leq\nu}(\delta)\cap\gamma)+z \leq \mathrm{otp}(D_{\leq\nu}(\delta))+z < \mathrm{otp}(D_{\leq\nu}(\delta))+\omega.$$

□

Define a function $f : A \to \omega_1$ by stipulating:

$$f(\delta) := \mathrm{otp}(D_{\leq\nu}(\delta^*)).$$

Claim 5.26.3. *Let $\beta < \delta$ be a pair of ordinals from A. Then $f(\beta) < f(\delta)$.*

Proof. Denote $\gamma := \beta^*$. Since $\delta \in E^{\omega_2}_{\omega_1}$ and since γ is the first element of $E^{\omega_2}_{\omega_1}$ above β, we have $\beta < \gamma \leq \delta$. It immediately follows:

$$f(\beta) = \mathrm{otp}(D_{\leq\nu}(\gamma)) < \mathrm{otp}(D_{\leq\nu}(\delta)) + \omega < \mathrm{otp}(D_{\leq\nu}(\delta^*)) = f(\delta).$$

□

So f forms an injection from A to ω_1, contradicting the fact that $|A| = \aleph_2$. □

The preceding theorem raises the following question. Does there exist a function $\varrho : [\omega_2]^2 \to \omega_1$ which is locally small and ω_1-coherent?

It turns out that the answer is affirmative, and moreover, for every infinite regular cardinal λ, for a suitable choice of a ladder system over λ^+, the corresponding function $\rho_1 : [\lambda^+]^2 \to \lambda$ is indeed locally small and λ-coherent. Out next goal is to show that, in general, it is hopeless to expect that this will generalize to λ singular.

Recall that, for an infinite cardinal κ, a *κ-Aronszajn tree* is a downward-closed family $\mathcal{T} \subseteq {}^{<\kappa}\kappa$ such that:

- $\{\mathrm{dom}(t) \mid t \in \mathcal{T}\} = \kappa$;
- for all $\delta < \kappa$, $\mathcal{T}_\delta := \mathcal{T} \cap {}^{\delta}\kappa$ has size $< \kappa$,
- for all $b : \kappa \to \kappa$, there is $\alpha < \kappa$ with $b \restriction \alpha \notin \mathcal{T}$.

A λ^+-Aronszajn tree is said to be *special* iff it is the union of λ-many antichains.

Definition 6.1 (Lücke, 2018)**.** For infinite cardinals $\theta < \kappa$ and a κ-Aronszajn tree $\mathcal{T}$, a sequence $\langle a_\alpha \mid \alpha \in H\rangle$ is a *θ-ascending path* iff all of the following hold true:

- H is a cofinal subset of κ;
- for every $\alpha \in H$, a_α is a function from θ to $\mathcal{T}_\alpha$;
- for every pair $\alpha < \beta$ of ordinals of H, there exist $i, j < \theta$ such that $a_\alpha(i)$ is below $a_\beta(j)$.

Exercise 6.2. *Prove that a κ-tree admits a θ-ascending path iff it admits one in which $H = \kappa$.*

Note that every λ^+-Aronszajn tree admits a λ-ascending path.

Exercise 6.3 (Baumgartner-Malitz-Reinhardt, 1970)**.** *Prove that any $\aleph_1$-Aronszajn tree admits no n-ascending path for any integer $n > 0$.*

Exercise 6.4 (Shelah-Stanely, 1988)**.** *Prove that any $\aleph_2$-Aronszajn tree that admits an $\aleph_0$-ascending path is not special.*

Definition 6.5. A filter $\mathcal{F}$ over a set X is a collection satisfying the following:

- $\mathcal{F} \subseteq \mathcal{P}(X)$;
- $X \in \mathcal{F}$, while $\emptyset \notin \mathcal{F}$;
- for all $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$;
- for all $A \subseteq B \subseteq X$, if $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

An illuminating example of a filter is the following

$$\mathcal{F}_1 := \{A \subseteq \mathbb{R} \mid \mathbb{R} \setminus A \text{ is a null set}\}.$$[1]

Definition 6.6. A filter $\mathcal{F}$ is said to be *κ-complete* if for any $\{A_i \mid i < \sigma\} \subseteq \mathcal{F}$ with $\sigma < \kappa$, we have $\bigcap_{i<\sigma} A_i \in \mathcal{F}$.

For instance, $\mathcal{F}_1$ is $\aleph_1$-complete, since the countable union of null sets is null.

[1]Recall that $A \subseteq \mathbb{R}$ is said to be a *null set* if for every $\epsilon > 0$, there exists a countable sequence of open intervals $\langle I_n \mid n < \omega\rangle$ such that $A \subseteq \bigcup_{n<\omega} I_n$, and the sum $\sum_{n<\omega} \mathrm{Diam}(I_n)$ of the diameters of the I_n's is $< \epsilon$.

i.e. A finite intersection is still in the filter.

Exercise 6.7. *Prove that any filter is $\aleph_0$-complete.*

Definition 6.8. A filter $\mathcal{F}$ is said to be uniform if, for all $A, B \in \mathcal{F}$, $|A| = |B|$.

For instance, for any infinite cardinal θ, $\{X \subseteq \theta \mid |\theta \setminus X| < \theta\}$ is uniform and $\mathrm{cf}(\theta)$-complete.

Definition 6.9. A filter $\mathcal{F}$ over some set X is said to be an *ultrafilter*, if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $(X \setminus A) \in \mathcal{F}$.

Lemma 6.10 (Pigeonhole principle for ultrafilters)**.** *Suppose that $\mathcal{U}$ is a κ-complete ultrafilter.*

For any $A \in \mathcal{U}$, and any function $f : A \to \theta$ with $\theta < \kappa$, there exists some $B \subseteq A$ with $B \in \mathcal{U}$ on which f is constant.

Proof. For each $i < \theta$, let $A_i := \{a \in A \mid f(a) = i\}$. If there exists some $i < \theta$ such that $A_i \in \mathcal{U}$, then we are done.

Otherwise, since $\mathcal{U}$ is an ultrafilter, we infer that $A \setminus A_i$ is in $\mathcal{U}$ for all $i < \theta$. Since $\mathcal{U}$ is κ-complete, then, $\emptyset = A \setminus \bigcup_{i<\theta} A_i = \bigcap_{i<\theta}(A \setminus A_i)$ is in $\mathcal{U}$. This is a contradiction. $\square$

By Zorn's lemma, any filter $\mathcal{F}$ over a set X may be extended to an ultrafilter $\mathcal{U}$ over X, so that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{P}(X)$. However, in general, the ultrafilter $\mathcal{U}$ may have smaller completeness degree than that of $\mathcal{F}$.

Definition 6.11. A cardinal κ is said to be *strongly compact* iff κ is regular, uncountable, and, for every set X, any κ-complete filter $\mathcal{F}$ over X may be extended to a κ-complete ultrafilter $\mathcal{U}$ over X.

Definition 6.12. A cardinal κ is said to be *measurable* iff is is uncountable, and there exists a uniform κ-complete ultrafilter $\mathcal{U}$ over κ.

Recall that a cardinal κ is *inaccessible* iff it is a limit cardinal which is regular and uncountable.

Definition 6.13. A cardinal κ is said to be *weakly compact* iff it is inaccessible, and there exist no κ-Aronszajn trees.

Exercise 6.14. *Prove: for every cardinal κ, (1) $\implies$ (2) $\implies$ (3) :*

(1) *κ is strongly compact;*
(2) *κ is measurable;*
(3) *κ is weakly compact.*

Exercise 6.15. *Prove that κ is weakly compact iff it is regular, uncountable, and Ramsey's theorem holds at the level of κ (i.e., for every coloring $c : [\kappa]^2 \to 2$, there exists $H \in [\kappa]^\kappa$ such that $c \restriction [H]^2$ is constant.)*

the case that T admits no θ -ascending path

Lemma 6.16. *Suppose that $\theta < \kappa \leq \lambda$ are infinite cardinals, κ is strongly compact, and λ is regular. Suppose $\mathcal{T}$ is a λ-Aronszajn tree. Then $\mathcal{T}$ admits no θ-ascending path.*

Proof. Towards a contradiction, suppose that $\langle a_\alpha \mid \alpha < \lambda\rangle$ is a θ-ascending path through $\mathcal{T}$. Define a coloring $c : [\lambda]^2 \rightarrow \theta \times \theta$ by letting, for all $\alpha < \beta < \lambda$,

$$c(\alpha, \beta) := \min\{(i, j) \in \theta \times \theta \mid a_\alpha(i) \text{ is below } a_\beta(j)\}.$$

Fix a κ-complete uniform ultrafilter $\mathcal{U}$ over λ. For all $\alpha < \lambda$ and $(i, j) \in \theta \times \theta$, write

$$A^{i,j}_\alpha := \{\beta \mid \alpha < \beta < \lambda, c(\alpha, \beta) = (i, j)\}.$$

Since $\mathcal{U}$ is uniform, $\biguplus_{(i,j)\in\theta\times\theta} A^{i,j}_\alpha = \lambda \setminus (\alpha + 1)$ is in $\mathcal{U}$. Since $\mathcal{U}$ is a κ-complete ultrafilter, we infer from Lemma 6.10 that there exists a pair $(i_\alpha, j_\alpha) \in \theta \times \theta$ such that $A^{(i_\alpha,j_\alpha)}_\alpha \in \mathcal{U}$.

Next, by the (usual) pigeonhole principle, let us fix $H \in [\lambda]^\lambda$ and $(i, j) \in \theta \times \theta$ such that, for all $\alpha \in H$, $(i_\alpha, j_\alpha) = (i, j)$. Now, for every pair $\alpha < \alpha'$ of ordinals from H, $A^{i,j}_\alpha \cap A^{i,j}_{\alpha'}$ is in $\mathcal{U}$, and we may pick some β in that intersection. It follows that $a_\alpha(i)$ and $a_{\alpha'}(i)$ are both below $a_\beta(j)$; as $\mathcal{T}$ is a tree, this means that $a_\alpha(i)$ is below $a_{\alpha'}(i)$. So $\bigcup_{\alpha\in H} a_\alpha$ is a branch through $\mathcal{T}$, contradicting the fact that it is Aronszajn. $\square$

The preceding result is optimal: in a paper by Brodsky and Rinot from 2017, it is shown to be consistent that for a strongly compact cardinal κ, and a regular cardinal λ above it, that there exists a λ-Aronszajn tree which admits a κ-ascending path.[2]

the case that T admits a θ -ascending path

Lemma 6.17. *Suppose that $\mathrm{cf}(\lambda) < \kappa < \lambda$ are infinite cardinals, with κ is strongly compact. Suppose $\mathcal{T}$ is a λ^+-Aronszajn tree. Then, for some cardinal $\theta < \lambda$, $\mathcal{T}$ admits a θ-ascending path.*

Proof. Fix a κ-complete uniform ultrafilter $\mathcal{U}$ over $\mathcal{T}$. For each $\alpha < \lambda^+$, fix a surjection $a_\alpha : \lambda \rightarrow \mathcal{T}_\alpha$. Fix a strictly increasing sequence $\langle\lambda_\eta \mid \eta < \mathrm{cf}(\lambda)\rangle$ of cardinals, converging to λ. For each $\alpha < \lambda^+$, define $f_\alpha : \mathcal{T} \setminus (\mathcal{T} \restriction \alpha) \rightarrow \mathrm{cf}(\lambda)$ by letting:

$$f_\alpha(t) := \min\{\eta < \mathrm{cf}(\lambda) \mid t \restriction \alpha \in a_\alpha[\lambda_\eta]\}.$$

Then, let η_α be the least $\eta < \mathrm{cf}(\lambda)$ such that

$$A^\eta_\alpha := \{t \in \mathrm{dom}(f_\alpha) \mid f_\alpha(t) = \eta\}$$

is in $\mathcal{U}$. Next, by the pigeonhole principle, fix $H \in [\lambda^+]^{\lambda^+}$ and $\eta < \mathrm{cf}(\lambda)$ such that, for all $\alpha \in H$, $\eta_\alpha = \eta$. Let $\theta := \lambda_\eta$. We claim that $\langle a_\alpha \restriction \theta \mid$

[2]In fact, λ-Souslin tree which admits a κ-ascent path.

$\alpha \in H\rangle$ is a θ-ascending path. To see this, fix an arbitrary pair $\alpha < \alpha'$ of ordinals from H. As $A^\eta_\alpha \cap A^\eta_{\alpha'}$ is in $\mathcal{U}$, we may pick some t in that intersection. Pick $i < \lambda_\eta$ such that $t \restriction \alpha = a_\alpha(i)$ and $j < \lambda_\eta$ such that $t \restriction \alpha' = a_{\alpha'}(j)$. Then, $i, j < \theta$ and $a_\alpha(i)$ is below $a_{\alpha'}(j)$. $\square$

Corollary 6.18 (Magidor-Shelah, 1995)**.** *Suppose that λ is the singular limit of strongly compact cardinals. Then there exist no λ^+-Aronszajn trees.*

Proof. Towards a contradiction, suppose that $\mathcal{T}$ is a λ^+-Aronszajn tree. Since there exists a strongly compact cardinal in between $\mathrm{cf}(\lambda)$ and λ, it follows from Lemma 6.17 that there exists a cardinal $\theta < \lambda$ such that $\mathcal{T}$ admits a θ-ascending path. Since there exists a strongly compact cardinal in between θ and λ, we get a contradiction to Lemma 6.16. $\square$

Definition 6.19. A coloring $c : [\lambda]^2 \to \theta$ is said to *subadditive* if the below two hold:

- *subadditivity of the first kind.* That is, for all $\alpha \leq \beta \leq \gamma < \theta$:
$$c(\alpha, \gamma) \leq \max\{c(\alpha, \beta), c(\beta, \gamma)\};$$
- *subadditivity of the second kind.* That is, for all $\alpha \leq \beta \leq \gamma < \theta$:
$$c(\alpha, \beta) \leq \max\{c(\alpha, \gamma), c(\beta, \gamma)\}.$$

Note that subadditivity is equivalent to a coherence property of the derived filtration:

Exercise 6.20. *Given $c : [\lambda]^2 \to \theta$, for all $\gamma < \lambda$ and $\nu < \theta$, let us write $D_{\leq\nu}(\gamma) := \{\alpha < \gamma \mid c(\alpha, \gamma) \leq \nu\}$. Then the following are equivalent:*

(1) *c is subadditive;*
(2) *for all $\beta < \gamma < \lambda$ and $\nu < \theta$, if $\beta \in D_{\leq\nu}(\gamma)$, then*
$$D_{\leq\nu}(\beta) = D_{\leq\nu}(\gamma) \cap \beta.$$

Exercise 6.21. *Suppose that $c : [\lambda]^2 \to \theta$ is subadditive.*

For all $\alpha < \beta < \gamma < \lambda$, if $c(\alpha, \beta) > c(\beta, \gamma)$, then $c(\alpha, \gamma) = c(\alpha, \beta)$.

Lemma 6.22. *Suppose that θ is a limit nonzero ordinal, κ is strongly compact, λ is regular, and $\theta < \kappa \leq \lambda$.*

For every coloring $c : [\lambda]^2 \to \theta$ which is subadditive of the second kind, there exists $H \in [\lambda]^\lambda$ such that $\sup(c``[H]^2) < \theta$.

Proof. Fix a κ-complete uniform ultrafilter $\mathcal{U}$ over λ. Given a coloring c as above, for all $\alpha < \lambda$, let i_α be the least $i < \theta$ for which
$$A^i_\alpha := \{\beta \mid \alpha < \beta < \lambda, c(\alpha, \beta) = i\}$$

is in $\mathcal{U}$. Then, by pigeonhole principle, fix some $H \in [\lambda]^\lambda$ and $i < \theta$ such that $i_\alpha = i$ for all $\alpha \in H$.

We claim that $\sup(c``[H]^2) \leq i$. To see this, let $\alpha < \alpha'$ be an arbitrary pair of elements of H. By $A^{i_\alpha}_\alpha, A^{i_{\alpha'}}_{\alpha'} \in \mathcal{U}$ and $i_\alpha = i = i_{\alpha'}$, we may pick some $\beta \in A^i_\alpha \cap A^i_{\alpha'}$. Then $c(\alpha, \beta) = i = c(\alpha', \beta)$, so that subadditivity entails that $c(\alpha, \alpha') \leq i$. □

A recursive application of the preceding entails the following.

Exercise 6.23. *Suppose that κ is strongly compact, $\theta < \kappa \leq \lambda = \mathrm{cf}(\lambda)$.* i.e. λ is regular.

Prove that for every coloring $c : [\lambda]^2 \to \theta$ which is subadditive of the second kind, there exists $H \in [\lambda]^\lambda$ such that $c``[H]^2$ is finite.

Given a set I, we let $\mathcal{LO}_I$ denote the set of all linear orderings of I:

$$\mathcal{LO}_I := \{R \in \mathcal{P}(I \times I) \mid R \text{ is a total ordering of } I\}.$$

Note that any linear order $(L, \leq)$ of size κ admits a *representative* in $\mathcal{LO}_\kappa$, by simply taking a bijection $f : L \leftrightarrow \kappa$, and setting $R := \{(f(x), f(y)) \mid x \leq y\}$. So, in some sense, to understand all linear orderings, it suffices to study $\mathcal{LO}_\kappa$ for all cardinals κ.

For $R, S \in \mathcal{LO}_I$, write $R \trianglelefteq S$ if the ordered set (I, R) is embedded in (I, S), that is, if there exists an injection $f : I \to I$ such that $\{(f(x), f(y)) \mid (x, y) \in R\} \subseteq S$. Clearly, $\trianglelefteq$ is pre-order, that is, it is reflexive and transitive. A standard way to turn a pre-order into an order is to define an equivalence relation $\sim$ by letting $R \sim S$ iff $R \trianglelefteq S$ and $S \trianglelefteq R$, and then inherit an ordering on the quotient. However, the above equivalence relation does not express the concept that we are interested in. For instance, the unit interval $[0, 1]$ embeds into the half-open interval $(0, 1]$ via the map $x \mapsto \frac{x+1}{2}$, and $(0, 1]$ embeds into $[0, 1]$ via the identity map, while the representatives of the orderings of $[0, 1]$ and $(0, 1]$ in $\mathcal{LO}_\mathfrak{c}$ cannot be considered equivalent, since one has a minimal element and the other does not. Thus, instead, we shall define the equivalence relation $\sim$ by letting $R \sim S$ iff there exists a bijection $f : I \leftrightarrow I$ such that $S = \{(f(x), f(y)) \mid (x, y) \in R\}$. That is: $f(x)Sf(y)$ iff xRy.

This explains why use a bijection instead of two side pre-order to obtain a order.

cofinality is the cardinal of least cofinal subset.

Recall that the *cofinality* of a pre-order $(P, \leq)$ is the least size of a subset $A \subseteq P$ with the property that for all $x \in P$, there exists $y \in A$ with $x \leq y$. The co-initiality of $(P, \leq)$ is defined as the cofinality of $(P, \geq)$, where $x \geq y$ iff $y \leq x$.

Some natural questions that immediately come to mind concerning the structure $(\mathcal{LO}_\kappa, \trianglelefteq)$:

- What is the cofinality of $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- What is the co-initiality of $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- What are the sizes of maximal chains in $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- What are the sizes of maximal antichains in $(\mathcal{LO}_\kappa, \trianglelefteq)$?
- How many equivalence classes are in $(\mathcal{LO}_\kappa, \sim)$?

Exercise 7.1 (Cantor)**.** *Prove that any countable linear order embeds into the rationals* $(\mathbb{Q}, \leq)$.

Corollary 7.2. *The cofinality of* $(\mathcal{LO}_\omega, \trianglelefteq)$ *is* 1.

Exercise 7.3. *A linear order is said to be scattered it contains no copy of* $(\mathbb{Q}, \leq)$. *Compute the cofinality of the class of countable scattered orders.*

Definition 7.4. A linear order $(L, \leq)$ is said to be *separable*, if there exists a countable subset $D \subseteq L$, such that for all $a < b$ in L, there exists $d \in D$ with $a \leq d \leq b$.

i.e. it has a dense subset like the separable space.

Cantor's theorem implies:

Exercise 7.5. *For a linear order $(L, \preceq)$, the following are equivalent:*

(1) *$(L, \preceq)$ is separable;*
(2) *the real line $(\mathbb{R}, \leq)$ contains a copy of $(L, \preceq)$.*

Exercise 7.6. *$(\omega_1, \in)$ and $(\omega_1, \ni)$ do not contain an uncountable separable suborder.*

Exercise 7.7 (Sierpiński, 1933)**.** *A separable linear order does not contain a copy of $(\omega_1, \in)$ or of $(\omega_1, \ni)$.*

Definition 7.8. For a linearly ordered set $(L, \leq)$, write:

- $<:= \{(a, b) \in \leq \mid a \neq b\}$;
- $\leq^* := \{(b, a) \mid (a, b) \in \leq\}$;
- $L^2 := \{(a, b) \mid a \in L, b \in L\}$;
- $[L]^2 := \{(a, b) \in L^2 \mid a < b\}$;[1]
- $\leq^2$ is the relation over L^2 such that $(a, b) \leq^2 (c, d)$ iff $(a \leq c)$ and $(b \leq d)$.

Lemma 7.9. *The co-initiality of $(\mathcal{LO}_\omega, \lhd)$ is 2.*

Proof. Let $R := \in \restriction \omega^2$ and $S := R^*$, that is, $S = \ni \restriction \omega^2$. Given $T \in \mathcal{LO}_\omega$, define $c : [\omega]^2 \to 2$ by letting for all $n < m < \omega$: $c(n, m) = 1$ iff $(n, m) \in T$.

By Ramsey's theorem $\omega \to (\omega)^2_2$, we may find some an infinite $H \subseteq \omega$ which is c-homogeneous. Let $f : \omega \leftrightarrow H$ be the order-preserving bijection.

▶ If $c``[H]^2 = \{1\}$, then for all $n < m$, by $f(n) < f(m)$ in H, we have $c(f(n), f(m)) = 1$ and hence $(f(n), f(m)) \in T$. That is, $\{(f(n), f(m)) \mid (n, m) \in R\} \subseteq T$.

▶ If $c``[H]^2 = \{0\}$, then for all $n < m$, by $f(n) < f(m)$ in H, we have $c(f(n), f(m)) = 0$ and hence $(f(n), f(m)) \notin T$, and hence $(f(m), f(n)) \in T$. That is, $\{(f(n), f(m)) \mid (n, m) \in S\} \subseteq T$.

So $\{R, S\}$ is co-initial in $\mathcal{LO}_\omega$. In addition, (ω, S) does not embed in (ω, R) since the latter is well-ordered. Similarly, (ω, R) does not embed in (ω, S). So, the co-initiality of $(\mathcal{LO}_\omega, \trianglelefteq)$ is exactly 2. □

[1] This happens to be equivalent to the set $<$.

The reader may protest against the involvement of Ramsey's theorem in the preceding proof. However, we will see later that this is no incident, and there is a bilateral relation between Ramsey-type theorems for cardinals κ, and the co-initiality of $(\mathcal{LO}_\kappa, \trianglelefteq)$.

Exercise 7.10. *Exhibit a chain of order-type ω_1 in $(\mathcal{LO}_\omega, \trianglelefteq)$.*

Exercise 7.11. *Prove that $(\mathcal{LO}_\omega, \trianglelefteq)$ has an antichain of cardinality $2^{\aleph_0}$.*

Any uncountable subset X of ω_1 satisfies $(X, \in) \cong (\omega_1, \in)$ and $(X, \ni) \cong (\omega_1, \ni)$. In addition, by Exercise 7.5, any uncountable suborder of a separable linear order is a separable linear order. Is there another class of uncountable linear orders such that any uncountable suborder remains inside the class?

Definition 7.12. An uncountable linear order $(L, \leq)$ is said to be a *Countryman line* iff $(L^2, \leq^2)$ may be covered by countably many chains.

Exercise 7.13. *If $(C, \leq)$ is Countryman, then for every uncountable subset X of C, $(X, \leq)$ is Countryman.*

Exercise 7.14. *If $(C, \leq)$ is Countryman, then $(C, \leq^*)$ is Countryman.*

Observation 7.15. *If $(C, \leq)$ is Countryman, then $(C^2, \leq^2)$ is* ccc. *That is, it does not have uncountable antichains.*

Proof. Suppose that $C^2 \subseteq \bigcup_{n<\omega} A_n$, where $(A_n, \leq^2)$ is a chain for all $n < \omega$. If $Y \subseteq C^2$ is an antichain, then $|Y \cap A_n| \leq 1$ for all $n < \omega$, and hence Y is countable. $\square$

Corollary 7.16. *If $(C, \leq)$ is Countryman, then any linear-order that embeds to $(C, \leq)$ and to $(C, \leq^*)$ is countable.*

In particular, if $(\omega_1, \prec)$ is Countryman, then $\{\prec, \prec^*\}$ is an antichain in $\mathcal{LO}_{\omega_1}$.

Proof. Suppose that $(L, \preceq)$ is a linear order, and f is an embedding of $(L, \preceq)$ in $(C, \leq)$, and g is an embedding of $(L, \preceq)$ in $(C, \leq^*)$. Define $h : L \to C^2$ by letting $h(x) := (f(x), g(x))$. Let $x \neq y$ in L be arbitrary. Wlog, $x \prec y$. Then $f(x) < f(y)$ and $g(x) > g(y)$. It follows that range(h) is an antichain in $(C^2, \leq^2)$, and hence countable. As f is injective, so is h, and hence $|L| = |\text{range}(h)| \leq \aleph_0$. $\square$

Lemma 7.17 (Countryman, 1971)**.** *A Countryman line does not contain a copy of uncountable separable linear orders.*

Proof. Suppose that $(C, \leq)$ is a Countryman line, as witnessed by a partition $\{A_n \mid n < \omega\}$ of C^2 into $\leq^2$-chains. Suppose that $(L, \preceq)$ is an uncountable separable linear order that embeds to $(C, \leq)$ via a map $f : L \to C$. Let $D \subseteq L$ be a countable set witnessing that $(L, \preceq)$ is separable.

Claim 7.17.1. *For all $x \in L$, there exist $n < \omega$ and $d \in D$ for which the following holds. There exist $y, z \in L$ such that $y \prec d \prec z$ and $\{(f(x), f(y)), (f(x), f(z))\} \subseteq A_n$.*

Proof. Let $x \in L$ be arbitrary. Since $\{(f(x), f(y)) \mid y \in L\}$ is uncountable, let us pick some $n < \omega$ such that $\{(f(x), f(y)) \mid y \in L\} \cap A_n$ is uncountable. Pick $y_0 \prec y_1 \prec y_2 \prec y_3$ in L such that $(f(x), f(y_i)) \in A_n$ for all $i < 4$. Now, pick $d \in D$ such that $y_1 \preceq d \preceq y_2$. Then n and d works, as witnessed by $y := y_0$ and $z := y_3$. $\square$

For all $x \in L$, pick $(n_x, d_x) \in \omega \times D$ as in the preceding. Since $\omega \times D$ is countable, let us pick $x \prec x'$ in L for which $n_x = n_{x'}$, say n, and $d_x = d_{x'}$, say d. Pick $y \prec d \prec z$ and $y' \prec d \prec z'$ such that

$$\{(f(x), f(y)), (f(x), f(z)), (f(x'), f(y')), (f(x'), f(z'))\} \subseteq A_n.$$

In particular, the set on the left hand side of the above equation is linearly ordered by $\leq^2$. Since $x \prec x'$ and f is order-preserving, we have $f(x) < f(x')$ and hence $f(z) \leq f(y')$. Since f is order-preserving, then $z \preceq y'$, contradicting the fact that $y' \prec d \prec z$. $\square$

Corollary 7.18. *An uncountable separable linear order does not contain a copy of a Countryman line.*

Proof. Suppose not. By Exercise 7.5, this means that the real line $(\mathbb{R}, \leq)$ contains a copy of a Countryman line $C \subseteq \mathbb{R}$. Then, by Exercise 7.5 again, this means that $(C, \leq)$ is a separable Countryman line, contradicting Lemma 7.17. $\square$

In his 1971 paper, Roger Countryman conjectured that the separability hypothesis in Lemma 7.17 is surplus, i.e., that the now-called *Countryman lines* simply do not exist. However, in a paper from 1976, Shelah introduced a construction of such a linear order. We shall soon present a construction that uses walks on ordinals.

Lemma 7.19. *If $(C, \leq)$ is Countryman, then it does not contain a copy of $(\omega_1, \in)$.*

Proof. Let $\{A_n \mid n < \omega\}$ witness that $(C, \leq)$ is Countryman. Suppose that $f : \omega_1 \to C$ is an order-preserving function from $(\omega_1, \in)$ to $(C, <)$. For all $\alpha < \omega_1$, fix some $n_\alpha < \omega$ such that the set $\{(f(\alpha), f(\beta)) \mid \alpha < \beta < \omega_1\} \cap A_n$ is uncountable. By the pigeonhole principle, let us fix two ordinals $\alpha < \alpha' < \omega_1$ such that $n_\alpha = n_{\alpha'}$, say, it is n. Let:

- $B = \{\beta < \omega_1 \mid (f(\alpha), f(\beta)) \in A_n\}$;
- $B' = \{\beta' < \omega_1 \mid (f(\alpha'), f(\beta')) \in A_n\}$.

Then B and B' are uncountable, so we may find $\beta \in B$, $\beta' \in B'$ such that $\beta' < \beta$ and $\{(f(\alpha), f(\beta)), (f(\alpha'), f(\beta'))\} \subseteq A_n$. As $\alpha \in \alpha'$, we have $f(\alpha) < f(\alpha')$, so that $f(\beta) \leq f(\beta')$, contradicting the fact that f is order-preserving. $\square$

Corollary 7.20. *$(\omega_1, \in)$ does not contain a copy of a Countryman line.*

Proof. If L is an uncountable subset of ω_1 such that $(L, \in)$ is order-isomorphic to a Countryman line, then by $\mathrm{otp}(L, \in) = \omega_1$, this would mean that $(\omega_1, \in)$ itself is a Countryman line. However, the previous Lemma tells us that a Countryman line cannot contain a copy of $(\omega_1, \in)$. $\square$

Exercise 7.21.
1. *If $(C, \leq)$ is Countryman, then it does not contain a copy of $(\omega_1, \ni)$;*
2. *$(\omega_1, \ni)$ does not contain a copy of a Countryman line.*

Definition 7.22. Fix an arbitrary injection $f : \omega_1 \to \mathbb{R}$, and let $\leq_{\mathbb{R}} := \{(\alpha, \beta) \in (\omega_1)^2 \mid f(\alpha) \leq f(\beta)\}$. Then $\leq_{\mathbb{R}}$ is a representative in $\mathcal{LO}_{\omega_1}$ of some separable uncountable linear order.

Definition 7.23. Let $\leq_C \in \mathcal{LO}_{\omega_1}$ be such that $(\omega_1, \leq_C)$ is Countryman.[2]

Corollary 7.24. *$\mathcal{B}_5 := \{\in\restriction (\omega_1)^2, \ni\restriction (\omega_1)^2, \leq_{\mathbb{R}}, \leq_C, \leq_C^*\}$ is an antichain in $\mathcal{LO}_{\omega_1}$.*

The definition of $\leq_{\mathbb{R}}$ and $\leq_C$ seem somewhat odd, as the choice of representatives is completely arbitrary. However, we have the following interesting consistency results.

Fact 7.25 (Baumgartner, 1973)**.** *The proper forcing axiom (PFA) implies that $\leq_{\mathbb{R}}$ is co-initial in $\{R \in \mathcal{LO}_{\omega_1} \mid (\omega_1, R)$ is separable$\}$.*

Fact 7.26 (Moore, 2006)**.** *The proper forcing axiom (PFA) implies that $\mathcal{B}_5$ is co-initial in $\mathcal{LO}_{\omega_1}$.*

For completeness, let us also mention the following.

Fact 7.27 (Sierpiński, 1932)**.** *The continuum hypothesis implies that any co-initial set in $\{R \in \mathcal{LO}_{\omega_1} \mid (\omega_1, R)$ is separable$\}$ has size $2^{\aleph_1}$.*

Our next task is to present a construction of a Countryman line.

[2] We shall soon see it exists.

Definition 8.1 (Left lexicographic ordering)**.** For any countable set w and any $R \in \mathcal{LO}_w$, we derive a total ordering $\sqsubseteq_R$ on ${}^{<\omega_1}w$, as follows. For $s, t \in {}^{<\omega_1}w$, we distinguish two cases:

- if s and t are compatible, then simply let $s \sqsubseteq_R t$ iff $s \subseteq t$;
- if s and t are incompatible, write $\Delta(s,t) := \min\{\alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t) \mid s(\alpha) \neq t(\alpha)\}$, and let $s \sqsubseteq_R t$ iff $(s(\Delta(s,t)), t(\Delta(s,t))) \in R$.

s,t are compatible iff $\exists$x, st. x$\leqslant$s, x$\leqslant$t.

Exercise 8.2. *Prove that $\sqsubseteq_R$ is indeed a total ordering of ${}^{<\omega_1}w$.*

Definition 8.3. Let $\preceq_{\rho_1}$ be the element of $\mathcal{LO}_{\omega_1}$ which is inherited from $(\mathcal{T}(\rho_1), \sqsubseteq_\in)$ by identifying every ordinal $\alpha < \omega_1$ with the fiber $\rho_{1\alpha}$. That is, for $\alpha, \beta < \omega_1$, we let $\alpha \preceq_{\rho_1} \beta$ iff $\rho_{1\alpha} \subseteq \rho_{1\beta}$ or $\rho_{1\alpha}(\Delta(\alpha,\beta)) < \rho_{1\beta}(\Delta(\alpha,\beta))$.

Recall that an uncountable linear order $(L, \leq)$ is said to be a *Countryman line* iff $(L^2, \leq^2)$ may be covered by countably many chains.

Theorem 8.4. *$(\omega_1, \preceq_{\rho_1})$ is a Countryman line.*

Proof. We shall define a coloring $c : (\omega_1)^2 \to V_\omega$ such that the preimage of any singleton would make a $(\preceq_{\rho_1})^2$-chain.

We first dispose of a trivial case, letting $c(\delta,\delta) := 0$ for every $\delta < \omega_1$. Next, let $\gamma < \delta$ be arbitrary pair of countable ordinals; define:

- $\mathsf{d}(\gamma,\delta) := \{\tau < \gamma \mid \rho_{1\gamma}(\tau) \neq \rho_{1\delta}(\tau)\}$;
- $n_{\gamma,\delta} := \max\{0, \rho_{1\gamma}(\tau), \rho_{1\delta}(\tau) \mid \tau \in \mathsf{d}(\gamma,\delta)\}$;
- $E_{\gamma,\delta} := \{\tau \leq \gamma \mid \rho_{1\gamma}(\tau) \leq n_{\gamma,\delta} \text{ or } \rho_{1\delta}(\tau) \leq n_{\gamma,\delta}\}$.

Note that since ρ_1 is coherent, $n_{\gamma,\delta}$ is well-defined, and since ρ_1 is finite-to-one, $E_{\gamma,\delta}$ is finite. Clearly, $\mathsf{d}(\gamma,\delta) \subseteq E_{\gamma,\delta}$, and by our convention to let $\rho_1(\gamma,\gamma) = 0$ for all $\gamma < \omega_1$, we have $\max(E_{\gamma,\delta}) = \gamma$. Let:

- $m_{\gamma,\delta} := |E_{\gamma,\delta}|$;
- $\pi_{\gamma,\delta} : m_{\gamma,\delta} \leftrightarrow E_{\gamma,\delta}$ be the order-preserving bijection;
- $f_{\gamma,\delta} := \rho_{1\gamma} \circ \pi_{\gamma,\delta}$;
- $g_{\gamma,\delta} := \rho_{1\delta} \circ \pi_{\gamma,\delta}$;

Finally, let:

- $c(\gamma,\delta) = \{(0, n_{\gamma,\delta}), (1, m_{\gamma,\delta}), (2, f_{\gamma,\delta}), (3, g_{\gamma,\delta})\}$, and
- $c(\delta,\gamma) = \{(0, n_{\gamma,\delta}), (1, m_{\gamma,\delta}), (2, f_{\gamma,\delta}), (3, g_{\gamma,\delta}), (4,4)\}$.

Now, suppose that we are given ordinals $\alpha, \beta, \gamma, \delta < \omega_1$ such that $c(\alpha,\beta) = c(\gamma,\delta)$, say it is x. Note:

- If $x = 0$, then $\{(\alpha,\beta), (\gamma,\delta)\} = \{(\beta,\beta), (\delta,\delta)\}$ is a chain.
- If $\alpha = \gamma$, then $\{(\alpha,\beta), (\gamma,\delta)\} = \{(\alpha,\beta), (\alpha,\delta)\}$ is a chain.

- If $(4,4) \in x$, then $c(\beta,\alpha) = c(\delta,\gamma) = x\setminus\{(4,4)\}$ and $\{(\alpha,\beta),(\gamma,\delta)\}$ is a chain iff $\{(\beta,\alpha),(\delta,\gamma)\}$ is a chain.

Consequently, it suffices to prove that $\{(\alpha,\beta),(\gamma,\delta)\}$ is a chain, under the assumption that $\alpha \neq \gamma$ and x is of the form

$$x = \{(0,n),(1,m+1),(2,f),(3,g)\}.$$

In particular, $\alpha < \beta$ and $\gamma < \delta$.

Claim 8.4.1. $\mathsf{d}(\alpha,\gamma) \neq \emptyset$, *and* $\Delta(\alpha,\gamma) \notin E_{\alpha,\beta} \cap E_{\gamma,\delta}$.

Proof. By $\pi_{\alpha,\beta}(m) = \alpha \neq \gamma = \pi_{\gamma,\delta}(m)$, let $i \leq m$ be the least such that $\pi_{\alpha,\beta}(i) \neq \pi_{\gamma,\delta}(i)$.

► If $\pi_{\alpha,\beta}(i) < \pi_{\gamma,\delta}(i)$, then $\pi_{\alpha,\beta}(i) \in E_{\alpha,\beta}\cap\gamma\setminus E_{\gamma,\delta}$, and $\rho_{1\alpha}(\pi_{\alpha,\beta}(i)) \leq n < \rho_{1\gamma}(\pi_{\alpha,\beta}(i))$. In particular, $\Delta(\alpha,\gamma) \leq \pi_{\alpha,\beta}(i)$, so that $\mathsf{d}(\alpha,\gamma) \neq \emptyset$.

► If $\pi_{\gamma,\delta}(i) < \pi_{\alpha,\beta}(i)$, then $\pi_{\gamma,\delta}(i) \in E_{\gamma,\delta}\cap\alpha\setminus E_{\alpha,\beta}$, and $\rho_{1\gamma}(\pi_{\gamma,\delta}(i)) \leq n < \rho_{1\alpha}(\pi_{\gamma,\delta}(i))$. In particular, $\Delta(\alpha,\gamma) \leq \pi_{\gamma,\delta}(i)$, so that $\mathsf{d}(\alpha,\gamma) \neq \emptyset$.

Altogether, $\Delta(\alpha,\gamma) \leq \min\{\pi_{\alpha,\beta}(i),\pi_{\gamma,\delta}(i)\}$. Towards a contradiction, suppose that $\Delta(\alpha,\gamma) \in E_{\alpha,\beta} \cap E_{\gamma,\delta}$. Then there there must exist some $j < i$ such that $\pi_{\alpha,\beta}(j) = \Delta(\alpha,\gamma) = \pi_{\gamma,\delta}(j)$. However, in this case, we would get that $\rho_{1\alpha}(\Delta(\alpha,\gamma)) = f(j) = \rho_{1\gamma}(\Delta(\alpha,\gamma))$, which is a contradiction. □

Claim 8.4.2. $\Delta(\beta,\delta) \leq \Delta(\alpha,\gamma)$.

Proof. Note that $\Delta(\alpha,\gamma) < \alpha < \beta$ and $\Delta(\alpha,\gamma) < \gamma < \delta$. That is $\Delta(\alpha,\gamma) \in \alpha\cap\beta\cap\gamma\cap\delta$.

► If $\Delta(\alpha,\gamma) \notin E_{\alpha,\beta} \cup E_{\gamma,\delta}$, then $\rho_{1\beta}(\Delta(\alpha,\gamma)) = \rho_{1\alpha}(\Delta(\alpha,\gamma)) \neq \rho_{1\gamma}(\Delta(\alpha,\gamma)) = \rho_{1\delta}(\Delta(\alpha,\gamma))$. In particular, $\Delta(\beta,\delta) \leq \Delta(\alpha,\gamma)$.

► If $\Delta(\alpha,\gamma) \in E_{\alpha,\beta} \setminus E_{\gamma,\delta}$, then $\rho_{1\beta}(\Delta(\alpha,\gamma)) \leq n < \rho_{1\delta}(\Delta(\alpha,\gamma))$, and hence $\Delta(\beta,\delta) \leq \Delta(\alpha,\gamma)$.

► If $\Delta(\alpha,\gamma) \in E_{\gamma,\delta} \setminus E_{\alpha,\beta}$, then $\rho_{1\delta}(\Delta(\alpha,\gamma)) \leq n < \rho_{1\beta}(\Delta(\alpha,\gamma))$, and hence $\Delta(\beta,\delta) \leq \Delta(\alpha,\gamma)$. □

Without loss of generality, suppose that $\alpha \preceq_{\rho_1} \gamma$, and let us show that $\beta \preceq_{\rho_1} \delta$.

By Claim 8.4.1, the analysis splits into three cases:

► If $\Delta(\beta,\delta) \notin E_{\alpha,\beta} \cup E_{\gamma,\delta}$, then $\rho_{1\gamma}(\Delta(\beta,\delta)) = \rho_{1\delta}(\Delta(\beta,\delta)) \neq \rho_{1\beta}(\Delta(\beta,\delta)) = \rho_{1\alpha}(\Delta(\beta,\delta))$, and then $\Delta(\gamma,\alpha) \leq \Delta(\beta,\delta)$. So $\Delta(\beta,\delta) = \Delta(\alpha,\gamma)$, and $\rho_{1\delta}(\Delta(\beta,\delta)) = \rho_{1\gamma}(\Delta(\beta,\delta)) > \rho_{1\alpha}(\Delta(\beta,\delta)) = \rho_{1\beta}(\Delta(\beta,\delta))$, since $\alpha \preceq_{\rho_1} \gamma$. Thus, $\beta \preceq_{\rho_1} \delta$, as sought.

► If $\Delta(\beta,\delta) \in E_{\alpha,\beta}$, then for $i \leq m$ such that $\pi_{\alpha,\beta}(i) = \Delta(\beta,\delta)$, we have $\rho_{1\beta}(\Delta(\beta,\delta)) = g(i) = \rho_{1\delta}(\pi_{\gamma,\delta}(i))$, and hence $\pi_{\gamma,\delta}(i) \neq \Delta(\beta,\delta)$. Pick the least $j \leq i$ such that $\pi_{\alpha,\beta}(j) \neq \pi_{\gamma,\delta}(j)$. Then $\pi_{\alpha,\beta}(j) \in$

$E_{\alpha,\beta} \cap \gamma \setminus E_{\gamma,\delta}$,[1] and hence $\rho_{1\beta}(\pi_{\alpha,\beta}(j)) \leq n < \rho_{1\delta}(\pi_{\alpha,\beta}(j))$, and hence $\Delta(\beta,\delta) \leq \pi_{\alpha,\beta}(j) \leq \pi_{\alpha,\beta}(i) = \Delta(\beta,\delta)$, so that $j = i$.

It follows that $\Delta(\beta,\delta) \in E_{\alpha,\beta} \cap \gamma \setminus E_{\gamma,\delta}$, and hence $\rho_{1\beta}(\Delta(\beta,\delta)) \leq n < \rho_{1\delta}(\Delta(\beta,\delta))$, and hence $\beta \preceq_{\rho_1} \delta$, as sought.

▶ If $\Delta(\beta,\delta) \in E_{\gamma,\delta}$, then for $i \leq m$ such that $\pi_{\gamma,\delta}(i) = \Delta(\beta,\delta)$, we have $\rho_{1\delta}(\Delta(\beta,\delta)) = g(i) = \rho_{1\beta}(\pi_{\alpha,\beta}(i))$, and hence $\pi_{\alpha,\beta}(i) \neq \Delta(\beta,\delta)$. Pick the least $j \leq i$ such that $\pi_{\gamma,\delta}(j) \neq \pi_{\alpha,\beta}(j)$. Then $\pi_{\gamma,\delta}(j) \in E_{\gamma,\delta} \cap \alpha \setminus E_{\alpha,\beta}$,[2] and hence $\rho_{1\gamma}(\pi_{\gamma,\delta}(j)) \leq n < \rho_{1\alpha}(\pi_{\gamma,\delta}(j))$, and so $\Delta(\gamma,\alpha) \leq \pi_{\gamma,\delta}(j) \leq \pi_{\gamma,\delta}(i) = \Delta(\beta,\delta) \leq \Delta(\alpha,\gamma)$.

It follows that $\Delta(\alpha,\gamma) \in E_{\gamma,\delta} \cap \beta \setminus E_{\alpha,\beta}$, and hence $\rho_{1\alpha}(\Delta(\alpha,\gamma)) \leq n < \rho_{1\gamma}(\Delta(\alpha,\gamma))$, contradicting the fact that $\alpha \preceq_{\rho_1} \gamma$. So this case is void. □

The preceding proof only used the facts that ρ_1 is finite-to-one and coherent. An elaboration of this proof yields the following generalization.

Exercise 8.5. *If $\mathcal{T} \subseteq {}^{<\omega_1}\omega$ is an $\mathbb{R}$-embeddable coherent Aronszajn tree, then $(\mathcal{T}, \leq_\in)$ is Countryman.*

Exercise 8.6 (Peng, 2013)**.** *Prove that $(\mathcal{T}(\rho_2), \leq_\in)$ and $(\mathcal{T}_\pi(\rho_2), \leq_<)$ are isomorphic.*

Conclude that $(\omega_1, \preceq_{\rho_2})$ is Countryman.

So, we already have two examples of Countryman lines: one that is read from the special tree $\mathcal{T}(\rho_2)$, and one that is a read from a (consistently, nonspecial) Aronszajn tree $\mathcal{T}(\rho_1)$. But why does it have to be Aronszajn in the first place?

Definition 8.7. For a poset $(P, \leq)$, we let $\mathrm{Conv}(P, \leq)$ denote the collection of all nontrivial convex subsets of $(P, \leq)$, i.e., the collection of all $U \subseteq P$ of size > 1 with the property that for every $x \leq y \leq z$ in P: if $x, z \in U$, then $y \in U$.

Definition 8.8. A poset $(T, \leq_T)$ is a *tree* if for all $x \in T$, the set $x_\downarrow := \{y \in P \mid y <_T x\}$ is well-ordered by $\leq_T$. The α^{th}-level of the tree are all $x \in P$ such that $\mathrm{otp}(x_\downarrow, <_T) = \alpha$.

Definition 8.9 (The f-partition tree)**.** Given an infinite linear order $(L, \leq)$ and a choice function $f : \mathrm{Conv}(L, \leq) \to L$ such that $f(U) \neq \min(U)$ for all $U \in \mathrm{dom}(f)$, one defines $\mathcal{T}^f \subseteq \mathrm{Conv}(L, \leq)$ so that $(\mathcal{T}^f, \supseteq)$ would make a tree-ordering. This is done by recursion on $\alpha \in \mathrm{Ord}$:

[1] Note that $\pi_{\alpha,\beta}(i) = \Delta(\beta,\delta) \leq \Delta(\alpha,\gamma) < \gamma$. [recall Claim 8.4.2].

[2] Note that $\pi_{\gamma,\delta}(j) \leq \pi_{\gamma,\delta}(i) = \Delta(\beta,\delta) \leq \Delta(\alpha,\gamma) < \alpha$. [recall Claim 8.4.2].

- $T^f_0 := \{L\}$;
- $T^f_{\alpha+1} := \{\{y \in U \mid y < f(U)\}, \{y \in U \mid y \geq f(U)\} \mid U \in \mathcal{T}^f_\alpha\}\cap \operatorname{Conv}(L, \leq)$;
- $T^f_\alpha := \{\bigcap \operatorname{range}(b) \mid b \in \prod_{\beta<\alpha} T^f_\beta, (\operatorname{range}(b), \supseteq) \text{ is a chain}\} \cap \operatorname{Conv}(L, \leq)$ for limit nonzero α.

Finally, put $\mathcal{T}^f := \bigcup\{T^f_\alpha \mid \alpha \in \mathrm{Ord}\}$.

Notice that for every chain $\mathcal{B}$ in $(T^f, \supseteq)$, if $|\bigcap \mathcal{B}| \geq 2$, then $\bigcap \mathcal{B}$ is a non-terminal node in T^f.

Exercise 8.10. *Prove that for any ordinal α, T^f_α consists of pairwise disjoint sets.*

Exercise 8.11. *Prove that $T^f_\alpha \cap T^f_\beta = \emptyset$ whenever $\alpha \neq \beta$.*

It follows that $T^f_\alpha = \emptyset$ whenever $\alpha \geq (2^{|L|})^+$. The *height* of $\mathcal{T}^f$ is the least α for which $T^f_\alpha = \emptyset$.

Lemma 8.12. *If $(C, \leq)$ is a Countryman line, then for every choice function $f \in \prod \operatorname{Conv}(C, \leq)$ satisfying $f(U) \neq \min(U)$ for each $U \in \operatorname{dom}(f)$: $\mathcal{T}^f$ is an Aronszajn tree.*

Proof. Our proof will only make use of the fact that $(C, \leq)$ does not embed uncountable separable linear orders, nor $(\omega_1, \in)$ nor $(\omega_1, \ni)$.

Claim 8.12.1. $|T^f_\alpha| \leq \aleph_0$ *for all* $\alpha < \omega_1$.

Proof. Suppose not, and let $\alpha < \omega_1$ be the least for which $|T^f_\alpha| > \aleph_0$. For all $U \in T_\alpha$, pick $x_U \in U$. Now, if U, V are two distinct elements of T_α, then $x_U \neq x_V$, say $x_U < x_V$, and then by definition of the tree, the set $\{W \in \bigcup_{\beta<\alpha} T^f_\beta \mid \{x_U, x_V\} \subseteq W\}$ contains a maximal element with respect to the $\supseteq$-ordering. Let W be this maximal element. Then $x_U \leq f(W) \leq x_V$.

Thus, the countable set $\{f(W) \mid W \in \bigcup_{\beta<\alpha} T^f_\beta\}$ witnesses that the uncountable subset $\{x_U \mid U \in T_\alpha\}$ of C is separable. This is a contradiction. $\square$

Claim 8.12.2. $T^f_\alpha \neq \emptyset$ *for all* $\alpha < \omega_1$.

Proof. Suppose that T^f has a countable height, say δ. Then by the previous claim $D := \{f(U) \mid U \in T^f\}$ is countable. We shall reach a contradiction by showing that D witnesses that $(C, \leq)$ is separable.

Let $x < z$ be arbitrary elements of C. Let $\alpha \leq \delta$ be the least ordinal such that "$\{x, z\} \subseteq U$" does not hold for any $U \in T^f_\alpha$. Clearly, $\alpha > 0$. For all $\beta < \alpha$, there exists $U \in T^f_\beta$ such that $\{x, z\} \subseteq U$. So α must be a successor ordinal. Let $U \in T^f_{\alpha-1}$ be such that $\{x, z\} \subseteq U$.

▶ If $z < f(U)$, then $\{x, z\} \subseteq \{y \in U \mid y < f(U)\} \in T^f_\alpha$, contradicting the choice of α.

▶ If $x \leq f(U) \leq z$, then by $f(U) \in D$, we are good.

▶ If $f(U) < x$, then $\{x, z\} \subseteq \{x \in U \mid x \geq f(U)\} \in T^f_\alpha$, contradicting the choice of α. $\square$

Claim 8.12.3. *T^f does not have an uncountable branch. In particular, $T^f_{\omega_1}$ is empty.*

Proof. Suppose not. Pick $\langle U_\alpha \in T^f_\alpha \mid \alpha < \omega_1\rangle$ such that $\{U_\alpha \mid \alpha < \omega_1\}$ is linearly ordered by $\supseteq$. Define $g : \omega_1 \to 2$ by letting

$$g(\alpha) := \begin{cases} 1, & \text{if } U_{\alpha+1} = \{y \in U_\alpha \mid y \geq f(U_\alpha)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Pick an uncountable $Y \subseteq \omega_1$ on which g is constant.

▶ If $g[Y] = \{1\}$, then for every pair $\alpha < \beta$ of points from Y, we have

$$f(U_\beta) \in U_\beta \subseteq U_{\alpha+1} = \{y \in U_\alpha \mid y \geq f(U_\alpha)\},$$

so that $f(U_\beta) \geq f(U_\alpha)$. Since $f(U_\alpha)$ is the minimal element of $U_{\alpha+1} \supseteq U_\beta$, and $f(U_\beta) \neq \min(U_\beta)$, we moreover get that $f(U_\beta) > f(U_\alpha)$. Consequently, $\{f(U_\alpha) \mid \alpha \in Y\}$ is a copy of $(\omega_1, \in)$ in $(C, \leq)$, which is a contradiction.

▶ If $g[Y] = \{0\}$, then for every pair $\alpha < \beta$ of points from Y, we have

$$f(U_\beta) \in U_\beta \subseteq U_{\alpha+1} = \{y \in U_\alpha \mid y < f(U_\alpha)\},$$

and hence $f(U_\beta) < f(U_\alpha)$. Consequently, $\{f(U_\alpha) \mid \alpha \in Y\}$ is a copy of $(\omega_1, \ni)$ in $(C, \leq)$, which is again a contradiction. $\square$

This completes the proof. $\square$

Families of pairwise disjoint sets play a special rule in combinatorics. Unfortunately, often, such families are unavailable, and so, one is lead to settle for a slightly less nice families, called, Δ-*systems*. A Δ-system is a family of sets $\mathcal{A}$ for which there exist some auxiliary set r (the *root*) such that $a \cap b = r$ for all two distinct $a, b \in \mathcal{A}$. So a family of pairwise disjoint sets is the special case of a Δ-system with an empty root.

Lemma 9.1 (Δ-system lemma)**.** *Suppose that κ is a regular uncountable cardinal, $A \in [\kappa]^\kappa$, and $\{a_\alpha \mid \alpha \in A\}$ is an injective enumeration of finite subsets of κ.*

Then there exist $B \in [A]^\kappa$, $r \in [\kappa]^{<\omega}$, and $n < \omega$ such that:

- *$|a_\alpha| = n$ for all $\alpha \in B$;*
- *$a_\alpha \cap a_\beta = r$ for all $\alpha < \beta$ both from B;*
- *$\sup(r) < \alpha < \min(a_\delta \setminus r) \leq \max(a_\delta \setminus r) < \beta$ for all $\alpha < \delta < \beta$ in B.*

Proof. Since κ is regular and uncountable, there must exist some $n < \omega$ for which $\{\alpha \in A \mid |a_\alpha| = n\}$ has size κ. Thus, without loss of generality, we assume that $\alpha \mapsto |a_\alpha|$ is constant over A, with some value n. We now prove the lemma by induction on this n.

▶ For $n = 1$, we let $r := \emptyset$ and recursively determine a κ-sized subset $B \subseteq A \setminus \{0\}$ such that $\alpha < \min(a_\delta) = \max(a_\delta) < \beta$ for all $\alpha < \delta < \beta$ in B.

▶ Next, suppose that the claim holds for n, and let us prove it for $n + 1$.

▶▶ If $\{\min(a_\alpha) \mid \alpha < \omega_1\}$ has size κ, then we let $r := \emptyset$, and recursively determine a κ-sized subset $B \subseteq A \setminus \{0\}$ such that $\alpha < \min(a_\delta) < \max(a_\delta) < \beta$ for all $\alpha < \delta < \beta$ in B.

▶▶ Otherwise, there exists some $\varepsilon < \kappa$ such that $\{\alpha \in A \mid \min(a_\alpha) = \varepsilon\}$ has size κ. Now, appeal to the induction hypothesis with $\{a_\alpha \setminus \{\varepsilon\} \mid \alpha \in A\}$ to find $B \in [A]^\kappa$ and $r \in [\kappa]^{<\omega}$, so that B and $r \cup \{\varepsilon\}$ are as sought. □

Definition 9.2. Elements x, y of a poset $(P, \leq)$ are said to be *compatible* if there exists some $z \in P$ such that $x \leq z$ and $y \leq z$. $A \subseteq P$ is a *strong antichain* if any two distinct elements x, y of A are not compatible.[1]

Definition 9.3. A poset $(P, \leq)$ is said to be *κ-cc* if it has no strong antichains of size κ.

[1] That is, for no $z \in P$ do we have $x \leq z$ and $y \leq z$.

A topological space (X, τ) is said to be κ-cc if the poset $(\tau \setminus \{\emptyset\}, \supseteq)$ is κ-cc, that is, if any collection of pairwise disjoint open sets has size $< \kappa$. We know that the product of two separable spaces is again separable, and, more generally, that the product of two topological spaces of density κ is of density κ. What about the product of two κ-cc posets?

Exercise 9.4. *Prove that if there exist κ-cc posets $(P, \leq_P)$ and $(Q, \leq_Q)$ whose product is not κ-cc, then there exists a poset $(R, \leq)$ whose square $(R^2, \leq^2)$ is not κ-cc.*

Exercise 9.5 (Kurepa, 1952)**.** *If $(T, \leq)$ is a Souslin tree, then it is an example of an $\aleph_1$-cc poset whose square is not $\aleph_1$-cc.* \aleph_1-cc is also ccc.

Lemma 9.6. *Suppose that κ is a regular cardinal, and Ramsey's theorem holds at the level of κ, that is, $\kappa \rightarrow (\kappa)^2_2$. Then for any κ-cc poset $(P, \leq)$, the square $(P^2, \leq^2)$ is again κ-cc.*

Proof. Towards a contradiction, suppose that $(P, \leq)$ is κ-cc, and $A \subseteq P^2$ is a strong antichain of size κ. For all distinct $(x, x'), (y, y') \in A$, if there exists $z, z' \in P$ such that $x, y \leq z$ and $x', y' \leq z'$, then (z, z') would contradict the fact that (x, x') and (y, y') are incompatible. It follows that we may define a symmetric coloring $c : [A]^2 \rightarrow 2$ by letting $c((x, x'), (y, y')) = 0$ iff x and y are incompatible.

Now, by Ramsey's theorem for κ, we may find some $H \subseteq \kappa$ of size κ which is c-homogeneous.

► If $c``[H]^2 = \{0\}$, then $H_0 = \{x \mid (x, x') \in H\}$ is a strong antichain in $(P, \leq)$, and hence has size $< \kappa$. Since $|H| = \kappa$ and the latter is regular, there must exist some $x \in H_0$ such that $H^x := \{x' \mid (x, x') \in H\}$ has size κ. Since $(P, \leq)$ has the κ-cc, then we may find distinct $x', x'' \in H^x$ that are compatible. Then (x, x') and (x, x'') are compatible elements of A, which is a contradiction.

► If $c``[H]^2 = \{1\}$, then a similar argument would yield a contradiction. $\square$

It follows that if we would like to cook up a κ-cc poset whose square is not κ-cc, then we have to assume that Ramsey's theorem fails at κ. But how badly must it fail? The next definition provides a measure of the extent to which Ramsey's theorem fails at κ.

Definition 9.7 (Shelah, 1988)**.** $\mathrm{Pr}_1(\kappa, \theta, \chi)$ asserts the existence of a coloring $c : [\kappa]^2 \rightarrow \theta$ such that for every family $\mathcal{A} \subseteq [\kappa]^{<\chi}$ of κ many pairwise disjoint sets, and every color $\gamma < \theta$, there exists some $a, b \in \mathcal{A}$ with $\sup(a) < \min(b)$ such that $c[a \times b] = \{\gamma\}$.

Notice that Ramsey's theorem is equivalent to the failure of $\mathrm{Pr}_1(\omega, 2, 2)$.

Exercise 9.8. *Prove that if κ is a strongly-compact cardinal, then $\Pr_1(\kappa, \theta, n)$ fails for any $\theta < \kappa$ and positive $n < \omega$.*

Lemma 9.9. $\Pr_1(\kappa, 2, 2)$ *holds for every singular cardinal κ.*

Proof. Given a singular cardinal κ, let $\langle \kappa_i \mid i < \mathrm{cf}(\kappa)\rangle$ be an increasing sequence of cardinals, converging to κ. Define $d : \kappa \to \mathrm{cf}(\kappa)$ by letting $d(\alpha) := \min\{i < \mathrm{cf}(\kappa) \mid \alpha \leq \kappa_i\}$ for all $\alpha < \kappa$. Then define $c : [\kappa]^2 \to 2$ by stipulating $c(\alpha, \beta) := 1$ iff $d(\alpha) = d(\beta)$.

Let $H \subseteq \kappa$ of size κ be arbitrary.

► Since $d[H] \subseteq \mathrm{cf}(\kappa) < \kappa = |H|$, we know that $d \restriction H$ is not injective, and so there exist $\alpha < \beta$ in H such that $c(\alpha, \beta) = 0$.

► Since $d^{-1}\{i\}$ has size $\leq \kappa_i < \kappa$ for all $i < \mathrm{cf}(\kappa)$, there must exist $\alpha' < \beta'$ in H such that $c(\alpha, \beta) = 1$. □

Exercise 9.10 (Erdős-Tarski, 1943 + Kurepa, 1963)**.** *If κ is a singular strong limit cardinal, then the square of any κ-cc poset is again κ-cc.*

Theorem 9.11 (Galvin, 1980)**.** *If κ is a regular cardinal, and $\Pr_1(\kappa, 2, \omega)$ holds, then there exists a κ-cc poset whose square is not κ-cc.*

Proof. Let $c : [\kappa]^2 \to 2$ be a witness to $\Pr_1(\kappa, 2, \omega)$. Let P be the set of all pairs (x, i) where:

- x is a finite subset of κ;
- $i < 2$;
- $c``[x]^2 \subseteq \{i\}$.

We let $(x, i) \leq (y, j)$ iff $x \subseteq y$ and $i = j$.

The following is clear.

Claim 9.11.1. *For all $(x, i), (y, j) \in P$, (x, i) and (y, j) are compatible iff $i = j$ and $c``[x \cup y]^2 \subseteq \{i\}$.* □

Claim 9.11.2. $(P^2, \leq^2)$ *is not κ-cc.*

Proof. For every $\alpha < \kappa$, we have $(\{\alpha\}, i) \in P$ both for $i = 0$ and for $i = 1$, so let $A := \{((\{\alpha\}, 0), (\{\alpha\}, 1)) \mid \alpha < \kappa\}$. Clearly, A is a strong antichain of size κ in $(P^2, \leq^2)$. □

Claim 9.11.3. $(P, \leq)$ *is κ-cc.*

Proof. Suppose that we are given $A \subseteq P$ of size κ. By Ramsey's theorem, κ must be uncountable. So, by shrinking A, we may assume the existence of $n < \omega$ and $j < 2$ such that $|x| = n$ and $i = j$ for all $(x, i) \in A$. Let $\mathcal{A} = \{x \mid (x, j) \in A\}$. Then $\mathcal{A}$ has size κ, and by the Δ-system lemma, let us pick some $\mathcal{B} \subseteq \mathcal{A}$ of size κ, and a set r such that $x \cap y = r$ for all two distinct $x, y \in \mathcal{B}$. Let $\mathcal{B}' = \{x \setminus r \mid x \in \mathcal{B}\}$. Then $\mathcal{B}$ is a family of κ many pairwise disjoint finite subsets of κ. By the choice of c, let us pick $a, b \in \mathcal{B}$ such that $\sup(a) < \min(b)$ and $c[a \times b] = \{j\}$. Let $x := a \cup r$ and $y := b \cup r$. Then $(x, j), (y, j) \in A$. We shall reach a contradiction by verifying that the two are compatible. Indeed, we shall show that $c``[x \cup y]^2 = \{j\}$.

Let $\alpha < \beta$ be arbitrary elements of $x \cup y$.

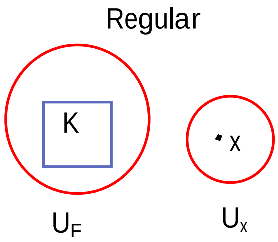
▶ If $\{\alpha, \beta\} \subseteq x$ or $\{\alpha, \beta\} \subseteq y$, then by $(x, j), (y, j) \in p$, we have $c(\alpha, \beta) = j$.

▶ If $\alpha \in x \setminus y$ and $\beta \in y \setminus x$, then $\alpha \in a$ and $\beta \in b$, and so by the choice of a and b, we have $c(\alpha, \beta) = j$.

▶ If $\alpha \in y \setminus x$ and $\beta \in x \setminus y$, then $\alpha \in b$ and $\beta \in a$, contradicting the fact that $\sup(a) < \min(b)$. □

This completes the proof. □

Exercise 9.12. *Prove that the forcing to add a Cohen real introduces* $\mathrm{Pr}_1(\omega_1, \omega_1, \omega)$.



Recall that a topological space $\mathbb{X} = (X, \tau)$ is said to be *Hausdorff* if for all two distinct points $x, y \in X$, there exist disjoint open sets U_x, U_y with $x \in U_x$ and $y \in U_y$. The space $\mathbb{X}$ is said to be *regular* if for every nonempty closed set F, and any point $x \notin F$, there exist disjoint open sets U_x, U_F with $x \in U_x$ and $F \subseteq U_F$. Equivalently, if for every point x and an open neighborhood U_x of x, there exists a closed subset $F \subseteq U_x$ where x is in its interior.

Definition 9.13 (Hajnal-Juhász, 1968)**.** An *L-space* is a regular Hausdorff topological space which is hereditarily Lindelöf but not separable.

An *S-space* is a regular Hausdorff topological space which is hereditarily separable but not Lindelöf.

Lindelof: every open cover has a countable subcover.
separable if it contains a countable, dense subset

A *strong L-space* (resp. *strong S-space*) is a topological space whose all finite powers are L-space (resp. S-space).

See Kunen & Vaughan, Handbook of Set-Theoretic Topology, pp. 303-304

Exercise 9.14 (Zenor, 1980)**.** *There is a strong L-space iff there is a strong S-space.*

We now consider a stronger form of Pr_1.

Definition 9.15 (Shelah, 1988)**.** $\mathrm{Pr}_0(\kappa, \theta, (\chi_0, \chi_1))$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every:

- $\sigma_0 < \chi_0$, and a family $\mathcal{A}_0 \subseteq [\kappa]^{\sigma_0}$ of size κ, consisting of pairwise disjoint sets;
- $\sigma_1 < \chi_1$, and a family $\mathcal{A}_1 \subseteq [\kappa]^{\sigma_1}$ of size κ, consisting of pairwise disjoint sets;
- prescribed coloring $g : \sigma_0 \times \sigma_1 \to \theta$,

there exists $(a_0, a_1) \in \mathcal{A}_0 \times \mathcal{A}_1$ with $\sup(a_0) < \min(a_1)$ such that
$$c(a_0(i), a_1(j)) = g(i,j) \text{ for all } (i,j) \in \sigma_0 \times \sigma_1.$$

Theorem 9.16 (Roitman?). *If $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$ holds, then there exists an L-space.*

Proof. Let $c : [\omega_1]^2 \to 2$ be a witness to $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$. We shall construct the L-space as a subspace of the product space ${}^{\omega_1}2$. The motivation comes from the following:

Claim 9.16.1. *Every subspace of ${}^{\omega_1}2$ is Hausdorff and regular.*

Proof. Recall that a basic open set in ${}^{\omega_1}2$ has the form $[s] = \{x \in {}^{\omega_1}2 \mid s \subseteq x\}$, where $s : a \to 2$ is a function from a finite subset of ω_1 to 2. Let $X \subseteq {}^{\omega_1}2$ be arbitrary.

▶ X is Hausdorff: if x, y are distinct points of X, then pick $\alpha < \omega_1$ such that $x(\alpha) \neq y(\alpha)$, and let $U_x := [\{(\alpha, x(\alpha)\}]$ and $U_y := [\{(\alpha, y(\alpha))\}]$. Then $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$.

▶ X is regular: if $x \in U_x$ and U_x is open, then by definition of the product topology, there exists some finite function s such that $x \in [s] \subseteq U_x$. For every $\alpha \in \mathrm{dom}(s)$, let $t_\alpha : \{\alpha\} \to 2$ be the unique function to satisfy $t_\alpha(\alpha) \neq s(\alpha)$. Then the complement of $[s]$ is equal to the finite union $\bigcup\{[t_\alpha] \mid \alpha \in \mathrm{dom}(s)\}$. This shows that $[s]$ is also closed. □

Let $X := \{x_\beta \mid \beta < \omega_1\}$, where for every $\beta < \omega_1$, $x_\beta : \omega_1 \to 2$ is defined by letting for all $\alpha < \omega_1$:
$$x_\beta(\alpha) := \begin{cases} c(\alpha, \beta), & \text{if } \alpha < \beta; \\ 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha > \beta. \end{cases}$$

Claim 9.16.2. *For every $\beta < \omega_1$, $\overline{\{x_\alpha \mid \alpha < \beta\}} \cap \{x_\gamma \mid \beta \leq \gamma < \omega_1\} = \emptyset$. In particular, X is not separable.*

Proof. Let $\beta < \omega_1$ be arbitrary. Let $\gamma \geq \beta$ be an arbitrary countable ordinal. We shall find an open neighborhood U of x_γ which is disjoint from $\{x_\alpha \mid \alpha < \beta\}$. Indeed, let $U := [s]$, where $s : \{\gamma\} \to \{1\}$ is a constant finite function. Then, $x_\gamma \in U$, while $\{x_\alpha \mid \alpha < \beta\} \cap U = \emptyset$. □

We remark that a space that can be well-ordered in such a way that every initial segment (according to this ordering) is closed — is said to be *left separated.*

WE SHALL CONTINUE WITH THE PROOF NEXT WEEK □

Recall that an *L-space* is a regular Hausdorff topological space which is hereditarily Lindelöf but not separable. An *S-space* is a regular Hausdorff topological space which is hereditarily separable but not Lindelöf.

Recall that $\mathrm{Pr}_0(\kappa, \theta, (\chi_0, \chi_1))$ asserts the existence of a coloring $c : [\kappa]^2 \to \theta$ such that for every:

- $\sigma_0 < \chi_0$, and a family $\mathcal{A}_0 \subseteq [\kappa]^{\sigma_0}$ of size κ, consisting of pairwise disjoint sets;
- $\sigma_1 < \chi_1$, and a family $\mathcal{A}_1 \subseteq [\kappa]^{\sigma_1}$ of size κ, consisting of pairwise disjoint sets;
- prescribed coloring $g : \sigma_0 \times \sigma_1 \to \theta$,

there exists $(a_0, a_1) \in \mathcal{A}_0 \times \mathcal{A}_1$ with $\sup(a_0) < \min(a_1)$ such that

$$c(a_0(i), a_1(j)) = g(i, j) \text{ for all } (i, j) \in \sigma_0 \times \sigma_1.$$

Theorem 10.1 (Roitman?). *If* $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$ *holds, then there exists an L-space.*

Proof. Let $c : [\omega_1]^2 \to 2$ be a witness to $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$. We shall construct the L-space as a subspace of the product space ${}^{\omega_1}2$. Let $X := \{x_\beta \mid \beta < \omega_1\}$, where for every $\beta < \omega_1$, $x_\beta : \omega_1 \to 2$ is defined by letting for all $\alpha < \omega_1$:

$$x_\beta(\alpha) := \begin{cases} c(\alpha, \beta), & \text{if } \alpha < \beta; \\ 1, & \text{if } \alpha = \beta; \\ 0, & \text{if } \alpha > \beta. \end{cases}$$

We already know that X is a regular Hausdorff space such having no uncountable separable space.

Claim 10.1.1. *If X is not hereditarily Lindelöf, then X contains an uncountable discrete space.*

Proof. Suppose that Y is a subspace of X which is not Lindelöf. Fix an open cover $\mathcal{U}$ of Y that does not admit a countable subscover. Since the weight of ${}^{\omega_1}2$ is ω_1, we may assume that $\mathcal{U} = \{U_i \mid i < \omega_1\}$.

Define functions $f : \omega_1 \to \omega_1$ and $g : \omega_1 \to \omega_1$ by stipulating:

- $f(\alpha) := \min\{\beta < \omega_1 \mid x_\beta \in Y \setminus \bigcup\{U_i \mid i < \alpha\}\}$;
- $g(\alpha) := \min\{i < \omega_1 \mid x_{f(\alpha)} \in U_i\}$.

Let $C := \{\gamma < \omega_1 \mid g[\gamma] \subseteq \gamma\}$. As $g(\alpha) \geq \alpha$ for all $\alpha < \omega_1$, f is weakly increasing, so that the range of $f \restriction C$ is uncountable. Pick an uncountable $A \subseteq C$ on which f is strictly increasing. We claim that $Z := \{x_{f(\alpha)} \mid \alpha \in A\}$ is discrete. To see this, fix an arbitrary $\alpha \in A$. Put

$$V_\alpha := U_{g(\alpha)} \setminus \overline{\{x_\beta \mid \beta < f(\alpha)\}}.$$

By Definition of $g(\alpha)$, $x_{f(\alpha)} \in U_{g(\alpha)}$. By a claim we proved last week, $x_{f(\alpha)} \notin \overline{\{x_\beta \mid \beta < f(\alpha)\}}$. So V_α is an open neighborhood of $x_{f(\alpha)}$. To see that $Z \cap V_\alpha$ is a singleton, let $\gamma \in A$ be arbitrary. We consider two cases:

▶ If $\gamma < \alpha$, then $f(\gamma) < f(\alpha)$, and by definition of V_α, we have $x_{f(\gamma)} \notin V_\alpha$.

▶ If $\alpha < \gamma$, then $g(\alpha) < \gamma$ and so by definition of $f(\gamma)$, $x_{f(\gamma)} \notin U_{g(\alpha)}$, let alone $x_{f(\gamma)} \notin V_\alpha$. □

Thus, to complete the proof that X is an L-space, we suppose that $\{x_\beta \mid \beta \in B\}$ is discrete, for some fixed uncountable $B \subseteq \omega_1$, and derive a contradiction.

As $\{x_\beta \mid \beta \in B\}$ is discrete, there exists a collection of finite functions $\{s_\beta : a_\beta \to 2 \mid \beta \in B\}$ such that $\{\alpha \in B \mid x_\alpha \in [s_\beta]\} = \{\beta\}$ for all $\beta \in B$. By passing to an uncountable subset $B' \subseteq B$ (via pigeonholes and the Δ-system lemma), we ensure the existence of $r \in [\omega_1]^{<\omega}$, $n < \omega$ and $f : n \to 2$ such that:

(1) $a_\alpha \cap a_\beta = r$ for all $\alpha < \beta$ in B';
(2) $\alpha < \min(a_\delta \setminus r) \leq \max(a_\delta \setminus r) < \beta$ for all $\alpha < \delta < \beta$ in B';
(3) $|a_\alpha| = n$ and $s_\alpha(a_\alpha(i)) = f(i)$ for all $i < n$ and $\alpha \in B'$.

Put $\mathcal{A} := \{a_\alpha \setminus r \mid \alpha \in B'\}$. Define $g : (n - |r|) \times 1 \to 2$ by stipulating:

$$g(i, 0) := f(i + |r|).$$

As $c : [\omega_1]^2 \to 2$ witnesses $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$, let us pick $a \in \mathcal{A}$ and $\beta \in B'$ with $\max(a) < \beta$ such that

$$c(a(i), \beta) = g(i, 0) \text{ for all } i < |a|.$$

Let $\alpha \in B'$ be such that $a = a_\alpha \setminus r$. Then $\max(a_\alpha) < \beta$ and for all $i < n - |r|$, we have:

$$x_\beta(a_\alpha(i + |r|)) = c(a_\alpha(i + |r|), \beta) = g(i, 0) = f(i + |r|) = s_\alpha(a_\alpha(i + |r|)).$$

In other words, we have:

$$x_\beta(a_\alpha(i)) = s_\alpha(a_\alpha(i)), \quad (|r| \leq i < n).$$

In parallel, by $\alpha, \beta \in B'$, we get from Clause (3) above that for all $i < |r|$:

$$s_\alpha(a_\alpha(i)) = f(i) = s_\beta(a_\beta(i)).$$

By Clauses (1) and (2) above, we infer that $a_\alpha(i) = r(i) = a_\beta(i)$ for all $i < |r|$.

As $x_\beta \in [s_\beta]$, it must be the case that $x_\beta(a_\beta(i)) = s_\beta(a_\beta(i))$ for all $i < n$, and hence

$$x_\beta(a_\alpha(i)) = s_\alpha(a_\alpha(i)), \quad (i < |r|).$$

Altogether, we have established that $x_\beta \restriction a_\alpha = s_\alpha$. So $x_\beta \in [s_\alpha]$ contradicting the fact that $\alpha \neq \beta$. $\square$

Exercise 10.2. *If $n < \omega$ and $\mathrm{Pr}_0(\omega_1, 2, (\omega, n+2))$ holds, then there exists an L-space X such that X^{n+1} is still an L-space.*

A *strong L-space* is a space all of whose finite powers are L.

Exercise 10.3 (Roitman, 1979)**.** *Show that after adding one Cohen real, $\mathrm{Pr}_0(\omega_1, \omega_1, (\omega, \omega))$ holds, and hence there is a strong L-space.*

A space that can be well-ordered in such a way that every initial segment (according to this ordering) is open — is said to be *right separated.*

Exercise 10.4. *If $\mathrm{Pr}_0(\omega_1, 2, (n+2, \omega))$ holds, then there exists a right separated subspace of ${}^{\omega_1}2$ whose n^{th}-power is an S-space.*

A *strong S-space* is a space all of whose finite powers are S-space.

Lemma 10.5 (Erdős)**.** *$\mathrm{Pr}_1(\omega_1, \omega, 2)$ is equivalent to $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$.*

Proof. Let $c : [\omega_1]^2 \to \omega$ be a witness to $\mathrm{Pr}_1(\omega_1, \omega, 2)$. For every $\beta < \omega_1$, fix a surjection $\psi_\beta : \omega \to \beta$. Define $d : [\omega_1]^2 \to \omega_1$ by letting $d(\alpha, \beta) := \psi_\beta(c(\alpha, \beta))$ for all $\alpha < \beta < \omega_1$. Then d witnesses $\mathrm{Pr}_1(\omega_1, \omega_1, 2)$. $\square$

Exercise 10.6. *Prove that the following are equivalent:*

(1) $\mathrm{Pr}_1(\omega_1, \omega, \omega)$*;*
(2) $\mathrm{Pr}_1(\omega_1, \omega_1, \omega)$*;*
(3) $\mathrm{Pr}_0(\omega_1, \omega_1, (\omega, \omega))$*.*

Our next goal is to construct an L-space from ZFC.

We shall now need a more informative variation of the function λ.

Definition 10.7 (Lower trace function)**.** Define $L : [\omega_1]^2 \to [\omega_1]^{<\omega}$, letting for $\alpha < \beta < \omega_1$:

$$L(\alpha, \beta) := \left\{ \max_{i \leq j} \sup(C_{\mathrm{Tr}(\alpha,\beta)(i)} \cap \alpha) \mid j < \rho_2(\alpha, \beta) \right\}.$$

Note that $L(\alpha, \beta)$ is a *set* of size $\leq \rho_2(\alpha, \beta)$. Note also that $\max(L(\alpha, \beta)) = \lambda(\alpha, \beta)$, and $\min(L(\alpha, \beta)) = \sup(C_\beta \cap \alpha)$.

The following definition is due to Moore, 2006:

Definition 10.8 (Oscillation of the lower trace)**.** Define $o_1 : [\omega_1]^2 \to \omega$ by letting for $\alpha < \beta < \omega_1$:

- $\tau^+ := \min(L(\alpha, \beta) \setminus \tau + 1)$ whenever $\tau < \lambda(\alpha, \beta)$;
- $o_1(\alpha, \beta) := |\{\tau \in L(\alpha, \beta) \cap \lambda(\alpha, \beta) \mid \rho_{1\alpha}(\tau) = \rho_{1\beta}(\tau) \ \&\ \rho_{1\alpha}(\tau^+) > \rho_{1\beta}(\tau^+) \ \&\ \tau^+ < \alpha\}|$.

Here, o_1 is the special case of a quite general concept:

$$\mathrm{Osc}(f,g,L) := |\{\tau \in L \cap \max(L) \mid f(\tau) = g(\tau) \ \&\ f(\tau^L) > g(\tau^L)\}|,$$

where $\tau^L := \min(L \setminus \tau + 1)$.

Out next task is to analyse o_1.

Definition 10.9. For a finite nonempty sets of ordinals L and U, write $A = L \oplus U$ whenever $A = L \cup U$ and $\max(L) < \min(U)$.

Lemma 10.10. *Suppose $\alpha < \beta < \delta < \omega_1$. If $\sup(C_\beta \cap \alpha) > \lambda(\beta,\delta)$, then:*

- $\mathrm{tr}(\alpha,\delta) = \mathrm{tr}(\beta,\delta)^\frown \mathrm{tr}(\alpha,\beta)$;
- $L(\alpha,\delta) = L(\beta,\delta) \oplus L(\alpha,\beta)$.

Proof. Since $\alpha \geq \sup(C_\beta \cap \alpha) > \lambda(\beta,\delta)$, we get from the second concatenation lemma that $\mathrm{tr}(\alpha,\delta) = \mathrm{tr}(\beta,\delta)^\frown \mathrm{tr}(\alpha,\beta)$, and hence $L(\alpha,\delta) = L(\beta,\delta) \cup U$, for

$$U := L(\alpha,\beta) \setminus (\max(L(\beta,\delta)) + 1).$$

Recalling that $\min(L(\alpha,\beta)) = \sup(C_\beta \cap \alpha) > \lambda(\beta,\delta) = \max(L(\beta,\delta))$, we conclude that $L(\alpha,\delta) = L(\beta,\delta) \oplus L(\alpha,\beta)$. □

Lemma 10.11. *Suppose that $m < \omega$ and $\mathcal{A} \subseteq [\omega_1]^m$ is a family of pairwise disjoint sets.*

Let $\Gamma(\mathcal{A})$ denote the set of all $\gamma < \omega_1$ such that for all

- $a \in \mathcal{A}$ *with* $\min(a) \geq \gamma$;
- $L \in [\gamma]^{<\omega}$;
- $U \in [\omega_1 \setminus \gamma]^{<\omega}$;
- $\eta < \omega$,

there exists some $a' \in \mathcal{A}$ satisfying the following:

(1) $\min(a') > \max(U)$;
(2) $\rho_1(\tau, a'(i)) > \eta$ *for all* $i < m$ *and* $\tau \in U$;
(3) $\rho_1(\tau, a'(i)) = \rho_1(\tau, a(i))$ *for all* $i < m$ *and* $\tau \in L$.

If $|\mathcal{A}| = \omega_1$, then $|\Gamma(\mathcal{A})| = \omega_1$.

Proof. Suppose that $\mathcal{A} \subseteq [\omega_1]^m$ consists of pairwise disjoint sets, and has size ω_1. For every $L \in [\omega_1]^{<\omega}$, $\varphi : L \times m \to \omega$, $\eta < \omega$ and $\varepsilon < \omega_1$, let $X(\varphi,\eta,\varepsilon)$ denote the set of all ordinals $\gamma < \omega_1$ such that there exists $a' \in \mathcal{A}$ satisfying:

- $\min(a') \geq \gamma$;
- $\rho_1(\tau, a'(i)) > \eta$ whenever $\varepsilon < \tau < \gamma$ and $i < m$;
- $\rho_1(\tau, a'(i)) = \varphi(\tau, i)$ whenever $\tau \in L$ and $i < m$.

Let

$$f(\varphi,\eta,\varepsilon) := \begin{cases} \sup(X(\varphi,\eta,\varepsilon)), & \sup(X(\varphi,\eta,\varepsilon)) < \omega_1 \\ \min(X(\varphi,\eta,\varepsilon) \setminus \varepsilon+1), & \text{otherwise} \end{cases}.$$

Define $g : \omega_1 \to \omega_1$ by stipulating:

$$g(\alpha) := \sup\{f(\varphi,\eta,\varepsilon) \mid L \in [\alpha]^{<\omega}, \varphi : L \times m \to \omega, \eta < \omega, \varepsilon < \alpha\}.$$

As $g(\alpha)$ is the sup over a set of size at most ω, g is well defined. Thus, consider the club $C := \{\gamma < \omega_1 \mid g[\gamma] \subseteq \gamma\}$, and let $\Gamma := \{\gamma \in C \mid \mathrm{cf}(\gamma) = \omega\}$. We shall show that $\Gamma \subseteq \Gamma(\mathcal{A})$.

Let $\gamma \in \Gamma$ be arbitrary. Fix $a \in \mathcal{A}$ with $\min(a) \geq \gamma$, $L \in [\gamma]^{<\omega}$, $U \in [\omega_1 \setminus \gamma]^{<\omega}$, and $\eta < \omega$. Define $\varphi : L \times m \to \omega$ by stipulating $\varphi(\tau,i) := \rho_1(\tau, a(i))$. As γ is a limit ordinal and the fibers of ρ_1 are finite-to-one, fix a large enough $\varepsilon < \gamma$ such that $\rho_1(\tau, a(i)) > \eta$ whenever $\varepsilon < \tau < \gamma$ and $i < m$. Let $\alpha := \max(L \cup \{\varepsilon\}) + 1$. By $\alpha < \gamma$ and $\gamma \in C$, we get that $f(\varphi,\eta,\varepsilon) \leq g(\alpha) < \gamma$. On the other hand, a witnesses that $\gamma \in X(\varphi,\eta,\varepsilon)$, and hence $f(\varphi,\eta,\varepsilon)$ is smaller than the sup of $X(\varphi,\eta,\varepsilon)$. That is, its sup is ω_1. Thus, pick $\delta \in X(\varphi,\eta,\varepsilon)$ above $\max(U)$, and then pick a witness $a' \in \mathcal{A}$ for the choice of δ. Then:

- $\min(a') \geq \delta > \max(U)$;
- $\rho_1(\tau, a'(i)) > \eta$ whenever $\varepsilon < \tau < \delta$ and $i < m$;
- $\rho_1(\tau, a'(i)) = \varphi(\tau,i) = \rho_1(\tau, a(i))$ whenever $\tau \in L$ and $i < m$.

Of course, the combination of the first two bullets imply that $\rho_1(\tau, a'(i)) > \eta$ for all $i < m$ and $\tau \in U$. So, we are done. □

Recall the definitions of interest.

Definition 11.1 (Lower trace function)**.** Define $L : [\omega_1]^2 \rightarrow [\omega_1]^{<\omega}$, letting for $\alpha < \beta < \omega_1$:

$$L(\alpha, \beta) := \left\{ \max_{i \leq j} \sup(C_{\mathrm{Tr}(\alpha,\beta)(i)} \cap \alpha) \mid j < \rho_2(\alpha, \beta) \right\}.$$

Definition 11.2 (Oscillation of the lower trace)**.** Define $o_1 : [\omega_1]^2 \rightarrow \omega$ by letting for $\alpha < \beta < \omega_1$:

- $\tau^+ := \min(L(\alpha, \beta) \setminus \tau + 1)$ whenever $\tau < \lambda(\alpha, \beta)$;
- $o_1(\alpha, \beta) := |\{\tau \in L(\alpha, \beta) \cap \lambda(\alpha, \beta) \mid \rho_{1\alpha}(\tau) = \rho_{1\beta}(\tau)\ \&\ \rho_{1\alpha}(\tau^+) > \rho_{1\beta}(\tau^+)\ \&\ \tau^+ < \alpha\}|$.

Definition 11.3. For a finite nonempty sets of ordinals L and U, write $A = L \oplus U$ whenever $A = L \cup U$ and $\max(L) < \min(U)$.

Lemma 11.4. *Suppose $\alpha < \beta < \delta < \omega_1$. If $\sup(C_\beta \cap \alpha) > \lambda(\beta, \delta)$, then $\mathrm{tr}(\alpha, \delta) = \mathrm{tr}(\beta, \delta)^\frown \mathrm{tr}(\alpha, \beta)$ and $L(\alpha, \delta) = L(\beta, \delta) \oplus L(\alpha, \beta)$.* □

Lemma 11.5. *Suppose that $m < \omega$ and $\mathcal{A} \subseteq [\omega_1]^m$ is a family of pairwise disjoint sets.*

Let $\Gamma(\mathcal{A})$ denote the set of all $\gamma < \omega_1$ such that for all

- *$a \in \mathcal{A}$ with $\min(a) \geq \gamma$;*
- *$L \in [\gamma]^{<\omega}$;*
- *$U \in [\omega_1 \setminus \gamma]^{<\omega}$;*
- *$\eta < \omega$,*

there exists some $a' \in \mathcal{A}$ satisfying the following:

1. *$\min(a') > \max(U)$;*
2. *$\rho_1(\tau, a'(i)) > \eta$ for all $i < m$ and $\tau \in U$;*
3. *$\rho_1(\tau, a'(i)) = \rho_1(\tau, a(i))$ for all $i < m$ and $\tau \in L$.*

If $|\mathcal{A}| = \omega_1$, then $|\Gamma(\mathcal{A})| = \omega_1$. □

Lemma 11.6. *Suppose that $m < \omega$, $\mathcal{A} \subseteq [\omega_1]^m$ is a family of pairwise disjoint sets, and $B \subseteq \omega_1$. If $|\mathcal{A}| = |B| = \omega_1$, then there exists a nonzero limit ordinal $\delta < \omega_1$ satisfying the following.*

For all $a \in \mathcal{A}$ and $\beta \in B$ with $\min(a), \beta > \delta$, there exist $a', a'' \in \mathcal{A}$, $\beta' \in B$, and $U \subseteq \delta$, for which all of the following hold:

1. *$L(\delta, \beta') = L(\delta, \beta) \oplus U$;*
2. *$\beta' > \delta$, and $\rho_{1\beta'} \restriction L(\delta, \beta) = \rho_{1\beta} \restriction L(\delta, \beta)$;*
3. *$\min(a') > \delta$, and $\rho_{1a'(i)} \restriction L(\delta, \beta) = \rho_{1a(i)} \restriction L(\delta, \beta)$ for all $i < m$;*
4. *$\min(a'') > \delta$, and $\rho_{1a''(i)} \restriction L(\delta, \beta) = \rho_{1a(i)} \restriction L(\delta, \beta)$ for all $i < m$;*
5. *$\rho_1(\tau, a'(i)) = \rho_1(\tau, \beta')$ for all $\tau \in U$ and $i < m$;*

(6) $\rho_1(\tau, a''(i)) > \rho_{1\beta'}(\tau, \beta')$ *for all* $\tau \in U$ *and* $i < m$*;*
(7) $\sup\{\min(\hat{a}) \mid \hat{a} \in \mathcal{A} \cap \mathcal{P}(\delta), \forall i < m\ \rho_{1\hat{a}(i)} \restriction L(\delta, \beta) = \rho_{1a(i)} \restriction L(\delta, \beta)\} = \delta$*.*

Proof. For every $L \in [\omega_1]^{<\omega}$, $\varphi : L \times m + 1 \to \omega$, and $\eta, \varepsilon < \omega_1$, let $X(\varphi, \eta, \varepsilon)$ denote the set of all ordinals $\delta' \in E^{\omega_1}_{\omega}$ such that there exists $(a', \beta') \in \mathcal{A} \times B$ satisfying:
(a) $\min(a'), \beta' > \delta'$;
(b) $\rho_1(\tau, \beta') = \varphi(\tau, m)$ whenever $\tau \in L$;
(c) $\rho_1(\tau, a'(i)) = \varphi(\tau, i)$ whenever $\tau \in L$ and $i < m$;
(d) $L(\delta', \beta') = L$;
(e) $\sup(C_{\delta'} \cap \eta) = \varepsilon$;
(f) $\rho_1(\tau, a'(i)) = \rho_1(\tau, \beta')$ whenever $\varepsilon \le \tau < \delta'$ and $i < m$.

Define $f(\varphi, \eta, \varepsilon)$ as in the proof of Lemma 11.5, and then derive the function $g : \omega_1 \to \omega_1$ by stipulating:

$$g(\alpha) := \sup\{f(\varphi, \eta, \varepsilon) \mid L \in [\alpha]^{<\omega}, \varphi : L \times m \to \omega, \eta, \varepsilon < \alpha\}.$$

Then there are club many $\delta < \omega_1$ such that $g[\delta] \subseteq \delta$ and $\sup(\Gamma(\mathcal{A}) \cap \delta) = \delta$. Thus, fix such a $\delta < \omega_1$, with $\mathrm{cf}(\delta) = \omega$.

Let $a \in \mathcal{A}$ and $\beta \in B$ be arbitrary, and satisfying $\min(a), \beta > \delta$.

As $\mathrm{cf}(\delta) = \omega$, we have $\lambda(\delta, \beta) < \delta$, so let us pick a large enough $\varepsilon \in \Gamma(\mathcal{A}) \cap \delta$ for which $\varepsilon > \lambda(\delta, \beta)$. As δ is a limit ordinal, we may also assume that $\varepsilon < \delta$ is so large that $\rho_1(\tau, a(i)) = \rho_1(\tau, \beta)$ whenever $\varepsilon \le \tau < \delta$ and $i < m$. Let $\eta := \min(C_\delta \setminus \varepsilon + 1)$, and

$$L := L(\delta, \beta).$$

Define $\varphi : L \times m + 1 \to \omega$ by letting for all $\tau \in L$:

$$\varphi(\tau, i) := \begin{cases} \rho_1(\tau, a(i)), & i < m \\ \rho_1(\tau, \beta), & i = m \end{cases}.$$

Let $\alpha := \max(L \cup \{\eta, \varepsilon\}) + 1$. Then $f(\varphi, \eta, \varepsilon) \le g(\alpha) < \delta$. As (a, β) witnesses that $\delta \in X(\varphi, \eta, \varepsilon)$, we infer that $X(\varphi, \eta, \varepsilon)$ is cofinal in ω_1. In particular, $f(\varphi, \eta, \varepsilon') < \delta$ whenever $\varepsilon \le \varepsilon' < \delta$, and hence we have verified Clause (7).

Next, pick $\delta' \in X(\varphi, \eta, \varepsilon)$ above δ, and let (a', β') be a witness to this choice.

By $L(\delta', \beta') = L = L(\delta, \beta)$, we have $\lambda(\delta', \beta') = \lambda(\delta, \beta) < \varepsilon = \sup(C_{\delta'} \cap \eta) < \eta < \delta < \delta' < \beta'$, and so Lemma 11.4 entails:

- $\mathrm{tr}(\delta, \beta') = \mathrm{tr}(\delta', \beta')^\frown \mathrm{tr}(\delta, \delta')$;
- $L(\delta, \beta') = L(\delta, \beta) \oplus L(\delta, \delta')$.

Denote

$$U := L(\delta, \delta').$$

Since δ is a limit ordinal, we have $\max(U) = \lambda(\delta, \delta') < \delta$. That is, $U \subseteq \delta$. Thus, we have established Clause (1). By $\beta' > \delta' > \delta$, $L(\delta', \beta') = L = L(\delta, \beta)$ and Clause (b), we have Clause (2). Likewise, Clause (c) entails Clause (3).

By $\min(U) = \sup(C_{\delta'} \cap \delta) \geq \sup(C_{\delta'} \cap \eta) = \varepsilon$ and $\lambda(\beta, \delta) < \varepsilon < \sup(C_{\delta'} \cap \eta) < \delta'$, we get from Clause (f) that Clause (5) holds.

Thus, we are left with finding a''. Let $U' := U \cup \{\delta\}$, $\eta' := \max\{\rho_{1\beta'}(\tau) \mid \tau \in U\}$. By $\varepsilon \in \Gamma(\mathcal{A})$, $\min(a) > \delta > \varepsilon$, $L \in [\varepsilon]^{<\omega}$, and $\min(U') \geq \varepsilon$, we may find some $a'' \in \mathcal{A}$ such that

(a) $\min(a'') > \max(U')$;
(b) $\rho_1(\tau, a''(i)) > \eta'$ for all $i < m$ and $\tau \in U'$;
(c) $\rho_1(\tau, a''(i)) = \rho_1(\tau, a(i))$ for all $i < m$ and $\tau \in L$.

By Clause (a), we have $\min(a'') > \delta$. By Clause (c), we get Clause (4), and by Clause (b) and the definition of η', we get Clause (6). □

Definition 11.7. A subset $T \subseteq \mathbb{N}$ is said to be *thick* if it contains arbitrary long intervals. That is, if for every positive $n < \omega$, there exists $k < \omega$ such that the integer interval $[k, k+n)$ is a subset of T.

Exercise 11.8. *There exists a function $d : \omega \to \omega$ such that $d``T = \omega$ for every thick subset $T \subseteq \omega$.*

Theorem 11.9. *Suppose that $m < \omega$, $\mathcal{A} \subseteq [\omega_1]^m$ is a family of pairwise disjoint sets, $B \subseteq \omega_1$, and $|\mathcal{A}| = |B| = \omega_1$. Then for every positive integer $n < \omega$, there exist $a \in \mathcal{A}$ and $\{\beta_0, \ldots, \beta_{n-1}\} \subseteq B$ such that:*

$$o_1(a(i), \beta_j) = o_1(a(i), \beta_0) + j, \quad (i < m, j < n).$$

Proof. Let $\delta < \omega_1$ be given by Lemma 11.6. In particular, there exist $a_0 \in \mathcal{A}$ and $\beta_0 \in B$ such that $\rho_1(\tau, a(i)) > \rho_1(\tau, \beta_0)$ for $\tau = \max(L(\delta, \beta_0))$ and $i < m$. Then by an iterative application of the choice of δ, we may find a sequence $\langle (a_{j+1}, \beta_{j+1}, E_j, G_j) \mid j \leq n \rangle$ such that for all $j \leq n$, the following hold:

(1) $L(\delta, \beta_{j+1}) = L(\delta, \beta_j) \oplus E_j \oplus G_j$;
(2) $\rho_{1\beta_{j+1}} \restriction L(\delta, \beta_j) = \rho_{1\beta_j} \restriction L(\delta, \beta_j)$;
(3) $\rho_{1a_{j+1}(i)} \restriction L(\delta, \beta_j) = \rho_{1a_j(i)} \restriction L(\delta, \beta_j)$ for all $i < m$;
(4) $\rho_{1a_{j+1}(i)}(\xi) = \rho_{1\beta_{j+1}}(\xi)$ for all $\xi \in E_j$ and $i < m$;
(5) $\rho_{1a_{j+1}(i)}(\xi) > \rho_{1\beta_{j+1}}(\xi)$ for all $\xi \in G_j$ and $i < m$.

By $\mathrm{cf}(\delta) = \omega$, fix a large enough $\gamma \in C_\delta$ with $\lambda(\delta, \beta_n) < \gamma$, and moreover

$$\gamma > \max\{\xi < \delta \mid \rho_{1\beta_j}(\xi) \neq \rho_{1\beta_{j+1}}(\xi) \text{ for some } j \leq n\}.$$

Write $L := L(\delta, \beta_n)$. By Clause (7) of the choice of δ, pick $\hat{a} \in \mathcal{A}$ with:

- $\gamma < \min(\hat{a}) \leq \max(\hat{a}) < \delta$;
- $\rho_{1\hat{a}(i)} \restriction L = \rho_{1a_n(i)} \restriction L$ for all $i < m$.

Claim 11.9.1. *For every $j \leq n$ and $i < m$, we have:*

(a) $\rho_{1\hat{a}(i)}(\max(L(\delta, \beta_j)) > \rho_{1\beta_j}(\max(L(\delta, \beta_j))$;
(b) $\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\delta, \beta_j)) = \mathrm{Osc}(\rho_{1a_j(i)}, \rho_{1\beta_j}, L(\delta, \beta_j))$.

Proof. Fix $j \leq n$. Then $L(\delta, \beta_j) \subseteq L(\delta, \beta_n) = L$, so by the choice of $\hat{a}$, we get that $\rho_{1\hat{a}(i)} \restriction L(\delta, \beta_j) = \rho_{1a_j(i)} \restriction L(\delta, \beta_j)$ for all $i < m$. $\square$

Note that item (a) of the preceding claim implies that for every $j \leq n, i < m$ and every finite $U \subseteq \delta$ with $\min(U) > \max(L(\delta, \beta_j))$, we have

$$\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\delta, \beta_j) \cup U) = \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\delta, \beta_j)) + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, U).$$

Claim 11.9.2. *For all $j \leq n$ and $i < m$ we have:*

(a) $L(\hat{a}(i), \beta_j) = L(\delta, \beta_j) \oplus L(\hat{a}(i), \delta)$;
(b) $\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \delta)) = \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_m}, L(\hat{a}(i), \delta))$;
(c) $\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_m)) = \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_m}, L(\delta, \beta_m))$.

Proof. Fix $j \leq n$. Note that the fact that $\min(\hat{a}) > \gamma \in C_\delta$, implies that $\min(L(\hat{a}(i), \delta)) = \max(C_\delta \cap \hat{a}(i)) \geq \gamma$. Now:

(a) follows from $\min(L(\hat{a}(i), \delta)) \geq \gamma > \lambda(\delta, \beta_j)$, and Lemma 11.4.

(b) follows from $\min(L(\hat{a}(i), \delta)) \geq \gamma > \max\{\xi < \delta \mid \rho_{1\beta_j}(\xi) \neq \rho_{1\beta_{j+1}}(\xi)\}$.

(c) follows from property (2) in the choice of $\langle (a_{j+1}(i), \beta_{j+1}, E_j, G_j) \mid m < \omega \rangle$. $\square$

Claim 11.9.3. $o_1(\hat{a}(i), \beta_{j+1}) = o_1(\hat{a}(i), \beta_j) + 1$ *for all $j < n$ and $i < m$.*

Proof. Fix $j < n$ and $i < m$. By the preceding claims, we get that

$$\begin{gathered}
o_1(\hat{a}(i), \beta_{j+1}) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \beta_{j+1})) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_{j+1}) \cup L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_{j+1})) + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_j) \cup E_j \cup G_j) + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_j)) + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, E_j \cup G_j) + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_j)) + \mathrm{Osc}(\rho_{1a_{j+1}(i)}, \rho_{1\beta_{j+1}}, E_j \cup G_j) + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\delta, \beta_j)) + 1 + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_{j+1}}, L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\delta, \beta_j)) + 1 + \mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\hat{a}(i), \delta)) = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\delta, \beta_j) \cup L(\hat{a}(i), \delta)) + 1 = \\
\mathrm{Osc}(\rho_{1\hat{a}(i)}, \rho_{1\beta_j}, L(\hat{a}(i), \beta_j)) + 1 = \\
o_1(\hat{a}(i), \beta_j) + 1.
\end{gathered}$$

□

So $o_1(\hat{a}(i), \beta_j) = o_1(a(i), \beta_0) + j$ for all $j < n$ and $i < m$, as sought.

□

We now turn to present the concept of a *b-universal sequence.*

Suppose for the moment that we are given a fixed sequence $\langle f_\alpha : \omega \to 2 \mid \alpha \in a\rangle$, indexed by some set a of ordinals. Then, for every function $h : a \to \omega$ and $i < \omega$, we define the i-evaluation of h, $h^i : a \to 2$, by letting for all $\alpha \in a$:

$$h^i(\alpha) := f_\alpha(h(\alpha) + i).$$

Given an *initial state* $h : a \to \omega$, it is quite natural to look at the induced orbit

$$\{h^i \mid i < \omega\} \subseteq {}^a2.$$

From there, we arrive to the concept of universality.

Definition 11.10. We say that a sequence $\langle f_\alpha : \omega \to 2 \mid \alpha < \omega_1\rangle$ is *universal*, if for every finite $a \subseteq \omega_1$, we have:

$$\{h^i \mid i < \omega\} = {}^a2 \text{ for all } h \in {}^a\omega.$$

An even more ambitious concept is that of *bounded* universality:

Definition 11.11. We say that a sequence $\langle f_\alpha : \omega \to 2 \mid \alpha < \omega_1\rangle$ is *b-universal*, if for every finite $a \subseteq \omega_1$, there exists $n(a) < \omega$, such that

$$\{h^i \mid i < n(a)\} = {}^a2 \text{ for all } h \in {}^a\omega.$$

Amazingly enough, this concept is feasible! To see this, first recall Kronecker's theorem on simultaneous diophantine approximation.

Fact 11.12 (Kronecker, 1884)**.** *Suppose that $a = \{x_0, \dots, x_{k-1}\}$ is a finite subset of $[0,1]$ such that $1, x_0, \dots, x_{k-1}$ are linearly independent over $\mathbb{Q}$. Then for every $\varepsilon > 0$, there exists $n(a,\varepsilon) \in \mathbb{N}$ such that for every $\vec{v} \in \mathbb{R}^k$, there exists a nonnegative integer $m < n(a,\varepsilon)$, and $\vec{z} \in \mathbb{Z}^k$ such that $|mx_i - \vec{v}(i) - \vec{z}(i)| < \varepsilon$ for all $i < k$.*

Theorem 11.13 (Moore, 2006)**.** *There exists a b-universal sequence.*

Proof. Recursively construct a sequence $\langle x_\alpha \mid \alpha < \omega_1\rangle$ of elements of $[0,1]$ such that $\{1\} \cup \{x_\alpha \mid \alpha \in a\}$ is linearly independent over $\mathbb{Q}$ for any finite $a \subseteq \omega_1$. Then, for every $\alpha < \omega_1$, define $f_\alpha : \omega \to 2$ by letting for all $n < \omega$:

$$f_\alpha(n) = 0 \text{ iff there exists some } z \in \mathbb{Z} \text{ such that } \left| nx_\alpha - \frac{1}{4} - z \right| < \frac{1}{8}.$$

To see that $\langle f_\alpha \mid \alpha < \omega_1\rangle$ is indeed b-universal, let $a \subseteq \omega_1$ be an arbitrary finite set.

Put $n(a) := n(\{x_\alpha \mid \alpha \in a\}, \frac{1}{8})$. Now, given an initial state $h : a \to \omega$, and a prescribed binary sequence $b : a \to 2$, we define $\vec{v} \in \mathbb{R}^a$ by letting for all $\alpha \in a$:

$$\vec{v}(\alpha) := \frac{1 + 2b(\alpha) - 4h(\alpha) \cdot x_\alpha}{4}$$

Pick $m \in \mathbb{N}, m < n(a)$, and $\vec{z} \in \mathbb{Z}^a$ such that $|mx_\alpha - \vec{v}(\alpha) - \vec{z}(\alpha)| < \frac{1}{8}$ for all $\alpha \in a$. Fix $\alpha \in a$:

► If $b(\alpha) = 0$, then $|mx_\alpha - \frac{1-4\cdot h(\alpha)\cdot x_\alpha}{4} - \vec{z}(\alpha)| < \frac{1}{8}$. That is, $|(m + h(\alpha))x_\alpha - \frac{1}{4} - \vec{z}(\alpha)| < \frac{1}{8}$, and hence $f_\alpha(h(\alpha)+m) = 0 = b(\alpha)$, as sought.

► If $b(\alpha) = 1$, then $|mx_\alpha - \frac{3-4\cdot h(\alpha)\cdot x_\alpha}{4} - \vec{z}(\alpha)| < \frac{1}{8}$. That is

$$\vec{z}(\alpha) + \frac{5}{8} < (m + h(\alpha))x_\alpha < \vec{z}(\alpha) + \frac{7}{8}.$$

Recalling that $f_\alpha(h(\alpha) + m) = 0$ iff there exists $z \in \mathbb{Z}$ such that

$$z + \frac{1}{8} < (m + h(\alpha))x_\alpha < z + \frac{3}{8},$$

we conclude that $f_\alpha(h(\alpha) + m) = 1 = b(\alpha)$, as sought. □ □

Meiri and Rinot found a purely combinatorial proof of the preceding result, and with the added feature of having a primitive-recursive bound for $n(a)$ for every $a \in [\omega_1]^{<\omega}$. The proof will appear elsewhere.

Theorem 11.14 (Moore, 2006)**.** $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$ *holds.*

In particular, there exists an L-space.

Proof. Let $\langle f_\alpha \mid \alpha < \omega_1\rangle$ be a b-universal sequence. Define $c : [\omega_1]^2 \to 2$ by stipulating $c(\alpha, \beta) := f_\alpha(o_1(\alpha, \beta))$. To see that c witnesses $\mathrm{Pr}_0(\omega_1, 2, (\omega, 2))$, suppose that $m < \omega$, $\mathcal{A} \subseteq [\omega_1]^m$ is an uncountable family of pairwise disjoint sets, and $B \subseteq \omega_1$ is also uncountable. Suppose that $g : m \to 2$ is a prescribed coloring.

Since $\langle f_\alpha \mid \alpha < \omega_1\rangle$ is b-universal, for every $a \in \mathcal{A}$, we have a corresponding positive integer $n(a)$. By thinning $\mathcal{A}$ out, we may assume the existence of $n < \omega$ such that $n(a) = n$ for all $a \in \mathcal{A}$.

By Theorem 11.9, then, there exist $a \in \mathcal{A}$ and $\{\beta_0, \ldots, \beta_{n-1}\} \subseteq B$ with $\max(a) < \beta_i$ for all $i < n$ such that:

$$o_1(a(i), \beta_j) = o_1(a(i), \beta_0) + j, \quad (i < m, j < n).$$

Define an initial state $h : a \to \omega$ by stipulating $h(\alpha) = o_1(\alpha, \beta_0)$. By $n = n(a)$, we know that

$$\{h^j \mid j < n\} = {}^a2.$$

Pick $j < n$ such that $h^j(a(i)) = g(i)$ for all $i < m$. Then, for all $\alpha \in a$:

$$h^j(\alpha) := f_\alpha(h(\alpha) + j) = f_\alpha(o_1(\alpha, \beta_0) + j) = f_\alpha(o_1(\alpha, \beta_j)) = c(\alpha, \beta_j).$$

Let $\beta := \beta_j$, then for all $i < m$:

$$c(a(i), \beta) = h^j(a(i)) = g(i),$$

as sought. □

By a slightly more elaborate proof than that of Theorem 11.13, Moore obtained the maximal number of colors:

Exercise 11.15 (Moore, 2006)**.** $\mathrm{Pr}_0(\omega_1, \omega_1, (\omega, 2))$ *holds.*

Recently, the latter was improved.

Exercise 11.16 (Peng-Wu, 2015)**.** *For all* $n < \omega$*, there is a* $\mathrm{Pr}_0(\omega_1, \omega_1, (\omega, n+2))$ *coloring.*